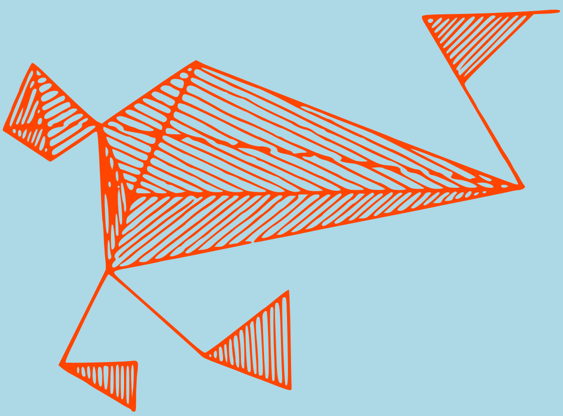
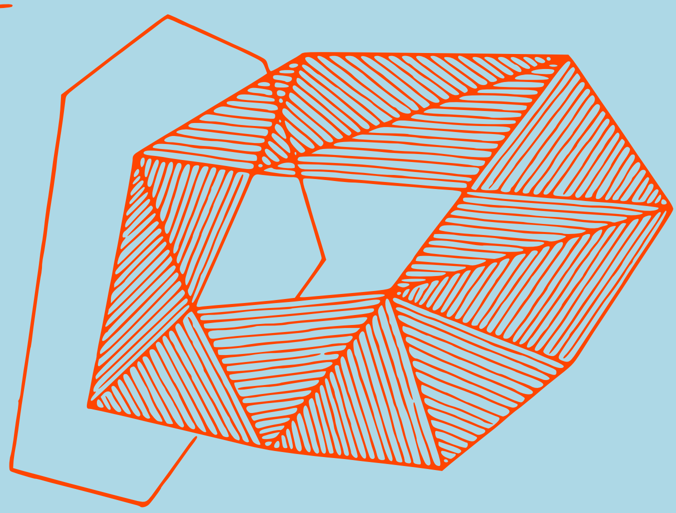
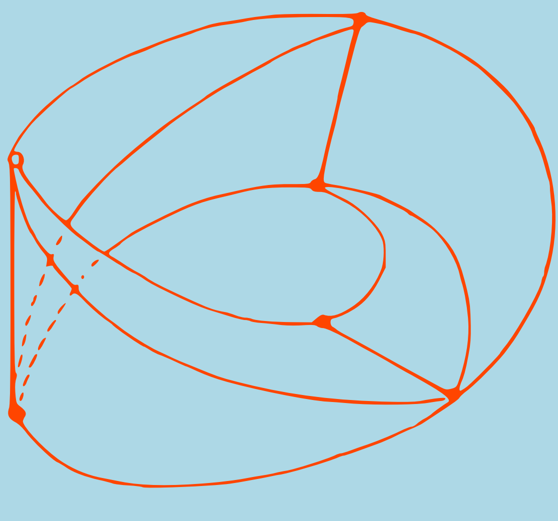
P.S. Aleksandrov

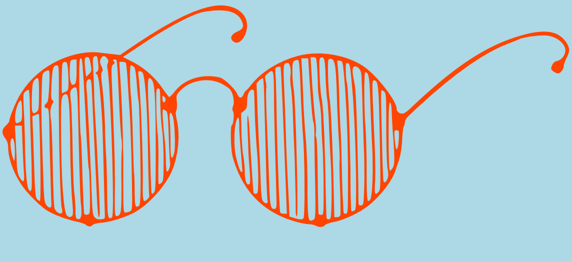
COMBINATORIAL TOPOLOGY

Volume 1









COMBINATORIAL TOPOLOGY
VOLUME 1

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COMBINATORIAL TOPOLOGY

VOLUME 1

BY

P. S. ALEKSANDROV

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TRANSLATOR'S NOTE

This volume is a translation of the first third of P. S. Aleksandrov's *Kombinatornaya Topologiya*. An appendix on the analytic geometry of Euclidean n-space is also included. The volume, complete in itself, deals with certain classical problems such as the Jordan curve theorem and the classification of closed surfaces without using the formal techniques of homology theory. The elementary but rigorous treatment of these problems, the introductory chapters on complexes and coverings and their applications to dimension theory, and the large number of examples and pictures should provide an excellent intuitive background for further study in combinatorial topology.

In Chapter I the references have been expanded to include a number of standard works in English. References to these and to the books and papers cited in Chapter I of the original are listed at the end of the chapter and correspond to the numbers enclosed in brackets in the body of the text. References in the remaining chapters are enclosed in brackets, capital letters referring to books and lower case letters to papers. These refer to the bibliography at the end of the book. The bibliography includes all papers mentioned in the original edition and a few which have been added by the translator. English translations are cited wherever possible. Cross-references to items in the text are made by citing chapter and section. Where the chapter number is omitted, the reference is to a section of the chapter being read. The system of transliteration used is that of the Mathematical Reviews. This may be confusing only in the cases of Aleksandrov and Tihonov, whose names are usually written in English as Alexandroff and Tychonoff.

CONTENTS

PART TWO

COMPLEXES. COVERINGS. DIMENSION

PREFACE

This book is an introduction to modern homology theory. It can be understood by anyone familiar with general set-theoretic and algebraic concepts and has been designed for the reader striving to acquire a knowledge of topology through a systematic study of its essentials. Hence, in this book, a reader can become acquainted with the ideas of modern topology *only by a detailed study* of the fundamental topological *facts*. I have endeavored to present these facts together with the necessary technical apparatus, often cumbersome and not at all attractive, with all logical rigor and at the risk of being boring and tiresome at times. The book can serve as a text for graduate students specializing in topology or any other branch of mathematics related to topology.

In writing this book I have made extensive use of *Topologie* I (see Alexandroff [A-H]), a joint work of the well known Swiss mathematician H. Hopf and myself. In particular, Chapter XVII of the present work is a translation, and Chapter XVI a revision, of Chapters XIV and XII, respectively, of *Topologie* I. I deem it especially necessary to emphasize this debt, since these two chapters of *Topologie* I were written in their entirety by H. Hopf. Both appendices of the present book (on Abelian groups and on the analytic geometry of n-dimensional space) are also in essence borrowed from the book of Alexandroff-Hopf, but are, in fact, considerably abridged. Besides these fundamental extracts, there are more superficial ones scattered throughout the text. It is hardly necessary to enumerate these individually.

Chapters VIII and IX are central to the whole book. They deal with what is called combinatorial homology theory. The necessary technical apparatus, complexes and chains, is presented in Chapters IV and VII. The approach in Chapter IV is combinatorial-geometric, while that in Chapter VII is algebraic. Chapters VIII and IX study not only the usual ("lower" or Δ-) homology but also the so called "upper" or ∇-homology (cohomology). The latter is constantly acquiring more and more significance in modern topological research. I have tried to expound the whole theory as simply and clearly as possible, elucidating it with a great many elementary examples whose assimilation is important for an understanding of the essential material.

The topological invariance of homology theory is proved in Chapters X and XI, which contain several different ways of proving the invariance of the Betti groups. In this connection I have completely dispensed with the so called "continuous" (singular) cycles, since it seems to me necessary to look to the "true" (proper) cycles, rather than to the singular cycles,

as the basic path of application of homology concepts to arbitrary compacta. True cycles are also convenient in practice in the study of polyhedra, in all cases where it is reasonable to use concepts whose very formulation is topologically invariant (e.g., in applications to the calculus of variations). In the principal questions of the construction of a general homology theory of compacta (particularly if one has in mind the generalization of this theory to the nonmetrizable case, first of all to arbitrary bicompacta) it is of course preferable to use "spectral theory" (developed in Chapter XIV), based on the direct investigation of the partially ordered set of all finite open or closed coverings of the space. For the study of the homology properties of a concrete individual compactum, however, true cycles defined in terms of the metric of the space are the most expedient.

A further development of the homology theory of modern topology is its *localization*, which reduces (by means of the notions of relative cycle and relative homology) to the definition of the Betti groups in a point. This makes it possible to study, e.g., manifolds with boundary, and also to develop the homology dimension theory. Of the latter we give in this book merely the definition of the homology dimension and a proof of the fact that, for polyhedra, this dimension is synonymous with the dimension in the elementary sense. Chapter XII is devoted to relative cycles and their applications.

At the present time we consider homology theory to be the fundamental core of topology because an extraordinarily significant number of the new geometric facts discovered in topology are formulated in terms of homology. Among the most important of these are the Poincaré and Alexander-Pontryagin dualities. Chapters XIII–XV are essentially devoted to these. The concept of homological manifold, the elements of the theory of intersections, as well as the theory of linking, also find their natural place here.

The latter theory, in its elementary form also turns out to be a convenient method for investigating the simplest classical questions of the theory of continuous mappings. Chapter XIV gives an account of this theory. Finally, the celebrated Lefschetz-Hopf formula for the algebraic number of fixed points of a continuous mapping of a polyhedron into itself is proved in Chapter XVII (borrowed in its entirety, as has already been said, from my joint work with Hopf).

The reader may choose from all the wealth of concrete factual material to which homology theory leads that which he finds interesting or useful. For example, one can, studying Chapter VIII and taking from Chapters X–XI any one of the proofs of the invariance of the Betti groups, pass at once to the theory of linking (Chapter XV) and then to the theory of continuous mappings (Chapters XVI and XVII), limiting oneself, if desirable,

to the classical topics presented in Chapter XVI. Or, one can, after Chapters VIII, IX, X, go on to Chapters XIII and XIV, i.e., basically, to the duality laws. Finally, the reader who wants to acquire the very elementary topological notions can, in more or less arbitrary order, read Chapters II, III, V; or, if the predominant interest is in questions of set-theoretic character, he can read Chapters V and VI, consulting the preceding chapters (basically Chapter IV) merely as a reference. Introductory remarks at the beginning of each part and chapter of the book will generally help the reader to choose a suitable sequence for the study of various topics.

Needless to say, this book does not begin to exhaust even the basic branches of modern combinatorial topology. Being oriented towards problems which are quite general and at the same time sufficiently elementary, this book, as is made clear above, seeks such problems in the domain of homology theory. The entire immense field of *homotopy methods*, from the classical fundamental group of Poincaré (see Seifert-Threlfall [S-T, Ch. 7]) to the homotopy groups of Hurewicz, and the recent research in homotopy problems of H. Hopf, Pontryagin, Eilenberg, and many others, remain entirely outside the scope of the book. Neither, however, have I touched on many deep questions of homology theory: the theory of intersections and products is represented merely by the wholly elementary case $p + q = n$, i.e., only when a zero-dimensional image is obtained in the intersection. It follows as a matter of course that all the results of the theory of continuous mappings which depend on the methods of "products" and "intersections" (Product Method of Lefschetz), as well as Hopf's extension of the theory, have been omitted. A series of papers which are due to appear in forthcoming topological issues of Uspehi Matematičeskih Nauk may be consulted on some of the topics mentioned (see Glezerman [a]). For a further development of questions related to the duality laws see Aleksandrov [f]. [I propose to devote a special monograph to questions of duality and homology dimension theory, in the hope of according these questions a fuller and more lucid treatment than they have received in the literature (see Aleksandrov [h, i, j]). Unfortunately, the methods developed in this paper, written after the completion of the book, could not be used here; otherwise, various improvements could have been made in Chapters XIII and XIV. I therefore recommend this paper especially as supplementary literature. Hurewicz-Wallman's *Dimension Theory* is recommended in the same vein.

The manuscript of this book was completed in the summer of 1941. Naturally, the outbreak of events delayed publication for a long time. After five years, as the book was about to appear, I was strongly tempted to consider many desirable changes in its text because of the advances in

Soviet and foreign scientific thought. I had to consider, first, the question of notation and terminology (see Aleksandrov [f]). However, each minor change in so unwieldy a work would have entailed others and would, in the end, have caused me to undertake a revision of the whole book, again delaying publication. I therefore decided to publish this manuscript in its original form and to postpone all possible improvements (whose necessity is more obvious to me than to anyone else) to a second edition.

In conclusion, I would like to thank all those who helped me in one way or another in the preparation of the book.

First of all, I would like to express my gratitude to L. S. Pontryagin for a close reading of the essential portions of the manuscript and for contributing a whole series of valuable remarks and suggestions. He frequently enabled me to improve the exposition substantially. I have noted in the text all cases of new and simpler proofs of propositions (as, e.g., in Chapters IV and XIII) communicated to me by L. S. Pontryagin. The continual friendly communication with L. S. Pontryagin over a long period of time has helped me a great deal in the preparation of the book and has been of great benefit to it. M. R. Šura-Bura has carefully read all of the book in proof, and I am obliged to him for many improvements in the exposition (especially in Chapter III).

I am obliged to A. N. Kolmogorov, A. S. Parhomenko, and A. M. Rodnyanskiĭ for a number of valuable remarks. I wish to thank A. N. Kolmogorov, in addition, for the execution of many of the figures.

My assistant U. Smirnov, a student at the University of Moscow, to whom I express here my sincere thanks, rendered me great assistance in the preparation of the final text of this book and a number of the figures.

I am indebted to S. V. Fomin for his help in reading proof and for compiling the index.

Finally, I am very grateful to the publishers (OGIZ) in the persons of its director G. F. Rybkin and the chief editor of mathematical literature A. I. Markuševič for their scrupulous attention to my wishes in connection with the publication of this book.

P. ALEKSANDROV

Bolševo, Komarovko
June 22, 1946

Part One

INTRODUCTION

Part One consists of three chapters. Chapter I is auxiliary in character. Chapters II and III are independent of each other and are concerned with the most elementary questions of "geometric" topology: the Jordan theorem and the elementary theory of closed surfaces. Chapters II and III, together with Chapter V, constitute a central core of interesting and important topological ideas. Without utilizing the general concepts of combinatorial topology, these chapters make the importance of such concepts quite clear.

Chapter I

PROPERTIES OF TOPOLOGICAL SPACES

Chapter I is an outline of the elementary theory of topological spaces. It is not meant for obligatory study. It may be used (after reading §1) merely as a reference. On the other hand, the student may read the chapter as part of the book and either furnish independent proofs where these are inadequate or omitted, or look up the proofs in, e.g., Hausdorff (see Translator's Note and [1]).

The results of Chapter I will not be applied often in the sequel, but we shall, of course, make systematic use of some of the concepts and propositions established here. In addition to such elementary ideas as continuous mapping, closed and open sets, etc., these include the concepts of deformation, Lebesgue numbers of a covering, partially ordered sets, and the theorem on the completeness of the metric space of continuous mappings of one compactum into another. The latter is applied in Chapter VI.

The notion of a bicompactum is used just once in the book, in the definition of the cohomology groups of bicompacta (see XIV). It would be perfectly possible to confine oneself here again to compacta (the reader may do so). Hence, all the elementary concepts introduced in Chapter I could have been introduced for metric spaces alone. The student can be guided by this suggestion as well in using Chapter I.

The exposition in this chapter is detailed in those cases which deal with concepts essential to the applications in the sequel. The same can be said of theorems cited here which are not to be found in Hausdorff (see Translator's Note and [1]). In the remaining cases the exposition has been abridged. Needless to say, all definitions and the statements of all theorems have been given in full.

§1. Notation of set theory

§1.1. Operations on sets. The *union* of the sets $A, B, C, \cdots$, i.e., the set of all elements contained in at least one of the sets $A, B, C, \cdots$, is denoted by $A \cup B \cup C \cdots$. Correspondingly, the union of the sets A_α, where α is an index (see 1.3) which assumes a finite or infinite number of values, is denoted by $\bigcup_\alpha A_\alpha$.

The *intersection* of the sets $A, B, C, \cdots$, denoted by $A \cap B \cap C \cdots$, is the set of elements contained in all of the sets $A, B, C, \cdots$. Similarly, $\bigcap_\alpha A_\alpha$ denotes the intersection of the sets A_α.

The *difference* of two sets A and B, denoted by $A \setminus B$, consists of those elements of the set A which are not elements of the set B. This definition does not presuppose that B is a subset of A, so that

$$A \setminus B = A \setminus (A \cap B).$$

The notation $A \subseteq B$, or $B \supseteq A$, means that the set A is a subset of the set B, i.e., that every element of A is an element of B.

$A \subset B$, or $B \supset A$, means that A is a *proper* subset of B, i.e., that every element of A is an element of B but that there is at least one element of B which is not an element of A.

The relation "a is an element of the set A" is written as $a \in A$.

The negation of the relations expressed by the symbols $\subseteq$, $\subset$, $\in$, and the like, is denoted by a line through these symbols. For example, $a \notin A$ means that a is not an element of A.

§1.2. Mappings. A *mapping* f of a set X *into* a set Y is an assignment to every element x of X of a definite element $y = f(x)$ of Y. The element $f(x)$ is called the *image* of the element x under the mapping f. If $A \subseteq X$, $f(A)$ is called the image of the set A under the mapping f and denotes the set of all elements in Y which are images of elements in A. If $B \subseteq Y$, the set of all $x \in X$ for which $f(x) \in B$ is called the *inverse image* of the set B under the mapping f and is denoted by $f^{-1}(B)$. If B, in particular, consists of a single element $y_0 \in Y$, $f^{-1}(B)$ is denoted by $f^{-1}(y_0)$ and is called the inverse image of the element y_0 under the mapping f. If the image of X under the mapping f is all of Y, f is said to be a mapping *onto* Y. The mapping f of the set X onto the set Y is said to be (1–1) if the inverse image of every element of Y under f consists of a single element of X.

Given mappings $f^1{}_2$ of a set X_1 into a set X_2 and $f^2{}_3$ of X_2 into a set X_3, assign to each $x_1 \in X_1$ the element $f^2{}_3[f^1{}_2(x_1)] \in X_3$. The result is a mapping

$$f^1{}_3 = f^2{}_3 f^1{}_2$$

of X_1 into X_3. Accordingly, for every $A_3 \subseteq X_3$,

$$(f^1{}_3)^{-1}(A_3) = (f^1{}_2)^{-1}[(f^2{}_3)^{-1}(A_3)].$$

REMARK. Instead of $f(x)$, $f(A)$, $f^{-1}(y)$, $f^{-1}(B)$, we will often write fx, fA, $f^{-1}y$, $f^{-1}B$.

§1.3. Indexed sets; systems of sets; order of a system of sets; coverings. Let f be a mapping of a set A into a set M [f need not be (1–1)] and denote a pair $\alpha \in A$, $m \in M$, where $f(\alpha) = m$, by m_α. It is sometimes convenient to call the set of such pairs "elements of M indexed by the set $\{\alpha\} = A$" or simply to say that M is indexed by A. The elements of M may, of course, themselves be sets. In that case, we have a family or system of sets indexed by A, or an indexed system of sets.

For instance, let $f(x, \theta)$ be a function of two variables, x and θ, defined for all pairs of values x, θ, $0 \leq x \leq 1$, $0 \leq \theta \leq 1$. Then for each fixed value of θ, the function

$$f_\theta(x) = f(x, \theta)$$

is a function of one variable x, and the functions $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ may coincide for two distinct values θ_1 and θ_2 of θ. The set of functions $f_\theta(x)$ is said to be indexed by θ, $0 \leq \theta \leq 1$.

Usually, instead of "a set indexed by A", we shall speak of a *family* or *system of elements* (which may themselves be sets) *depending on the index or parameter* α; for instance, we say that the family of functions $f_\theta(x)$ depends on the parameter θ.

Another important example is that of a system of subsets of a set R. Let us assign to each element i of a set of indices a subset A_i of the set R, where it is understood that distinct indices may correspond to identical subsets A_i, i.e., subsets consisting of the same elements. The resulting indexed system of subsets A_i of the set R is briefly referred to as a *system of subsets of R*.

A system of subsets of a set R is called a *covering* of R if the union of the sets of this system is all of R.

The systems and, in particular, the coverings considered in this book will consist almost exclusively of a finite number of sets. Accordingly, a basic notion will be the order of a covering. The *order* of a finite system of sets is the greatest integer n for which the system has n elements with nonempty intersection.

REMARK. A system of sets is said to be *simple* if every two elements of the system are distinct, i.e., if to every two distinct indices i and j correspond distinct sets A_i and A_j. We could, of course, consider only simple systems, but it is sometimes convenient not to be bound by this restriction. For examples of this see XIV.

§2. Topological spaces [2]

§2.1. Definition of topological spaces and basic related notions.

DEFINITION 2.11. A *topological space* is a set R composed of elements of arbitrary nature in which certain subsets $A \subseteq R$, called *closed sets* of the topological space R, have been defined so as to satisfy the following conditions, called the *axioms of a topological space*:

1_A. *The intersection of any number and the union of any finite number of closed sets is a closed set.*

2_A. *The whole set R and the empty set are closed sets.*

REMARK 1. The elements of the set R are called *points* of the topological space.

The sets complementary to the closed sets of R, i.e., the sets of the form $R \setminus A$, where A is closed, are called *open sets* of the topological space R. They clearly satisfy the following conditions:

1_Γ. *The union of any number and the intersection of any finite number of open sets is an open set.*

2_Γ. *The whole set R and the empty set are open.*

REMARK 2. A topological space could be defined as a set R in which certain subsets, called open sets, have been singled out and which satisfy 1_Γ–2_Γ. Then the sets complementary to the open sets satisfy 1_A–2_A and are said to be closed.

DEFINITION 2.12. The intersection of all closed sets containing a set M is called the *closure of M in the topological space R* and is denoted by $\bar{M}$. The points of $\bar{M}$ will be referred to as the *contact points of M*.

2.13. The *closure operation,* i.e., the passage from a set M to its closure $\bar{M}$, satisfies the following conditions:

$\bar{1}$. $\overline{M \cup N} = \bar{M} \cup \bar{N}$.

$\bar{2}$. $M \subseteq \bar{M}$.

$\bar{3}$. $\bar{\bar{M}} = \bar{M}$ *for arbitrary M.*

$\bar{4}$. *The closure of the empty set is empty.*

REMARK 3. A topological space could be defined as a set in which every subset M has been assigned a closure $\bar{M}$ satisfying $\bar{1} - \bar{4}$. Then closed sets are defined as sets which coincide with their closures. This leads to precisely the topological spaces which were defined at the beginning of this section.

DEFINITION 2.14. A *neighborhood* of a point p in a topological space R is any open set containing p.

A point $p \in M$ is called an *interior point* of the set M relative to the topological space R if p has a neighborhood contained in M. The set of all interior points of a set M, the *interior* of M, is an open set $\Gamma \subseteq M$. The set $\bar{M} \setminus \Gamma$ is called the *boundary* of M in R. A point of $\bar{M} \setminus \Gamma$ is a *boundary point* of M. A point of R is said to be *isolated* in R if the set consisting of this one point is open.

2.141. $p \in \bar{M}$ if, and only if, every neighborhood of the point p contains at least one point of the set M.

A point p is said to be a *point of accumulation* or a *limit point* of a set M if every neighborhood of p contains an infinite set of points of M.

DEFINITION 2.15. A set M is said to be *dense in R* if $\bar{M} = R$.

REMARK 4. A topological space is a composite of two concepts: the set of points of the topological space and the *topology defined in this set,* i.e., a system of sets exhibited as closed (open) sets of the topological space [or defined in the space by the closure operation (see Remark 3)].

2.16. Let R be a topological space and let $M \subset R$. The topology of R induces a topology in M (the *relative topology*) in the following way: *the sets closed in M are, by definition, the intersections with M of the closed sets of R.*

Whenever we refer to an arbitrary subset M of a topological space R as a topological space, we shall always have the relative topology in mind. This topology obviously satisfies 1_A–2_A.

It is easy to prove

2.17. The sets open in M are the intersections with M of the open sets of R.

§2.2. Neighborhood topology.

THEOREM 2.21. *Let R be any set and call its elements points. With each point $p \in R$ associate certain subsets $V(p)$ of R which contain p and are called neighborhoods of p in the given neighborhood system $\mathfrak{B}$. We shall assume that the given neighborhood system satisfies the following conditions:*

1_V. *The intersection of any two neighborhoods of a point $p \in R$ contains a neighborhood of p.*

2_V. *If q is any point contained in a neighborhood $V(p)$ of $p \in R$, there exists a neighborhood $V(q)$ of q contained in $V(p)$.*

Let us call a point p a contact point of a set M if every neighborhood of p contains at least one point of the set M, and let us define the closure $\bar{M}$ of M to be the set of all contact points of M. The closure operation defined in this way satisfies $\bar{1}$–$\bar{4}$ of 2.13 *and consequently converts the set R into a topological space in which all the neighborhoods $V(p)$ are open sets of R* (condition 2_V).

The collection of all neighborhoods of points of a topological space R defined in 2.14, i.e., the collection of all open sets of R, is conveniently referred to as the *absolute system of neighborhoods of R*; this system satisfies 1_V–2_V. The topology of the given topological space can therefore be defined by applying Theorem 2.21 to the absolute system of neighborhoods of the space, i.e., the neighborhood topology induced by using the system of open sets of R as the defining neighborhood system of 2.21 is the same as the original open set, closed set, or closure topology in R.

2.22. A collection $\mathfrak{B}$ of open sets Γ of a topological space R is called a *basis* for the space R if every open set of R is a union of sets of $\mathfrak{B}$.

It is clear that:

2.23. In order that a collection $\mathfrak{B}$ of open sets of a topological space R be a basis for R, it is necessary and sufficient that for every neighborhood V of an arbitrary point p (i.e., for every open set containing p) there exist an element Γ of $\mathfrak{B}$ such that

$$p \in \Gamma \subseteq V.$$

It follows from 2.141 that:

2.24. Let $\mathfrak{B}$ be a basis for R. Then p is a contact point of a set M if, and only if, every neighborhood of p, *which is at the same time an element of* $\mathfrak{B}$, contains at least one point of M.

DEFINITION 2.25. Let R be a topological space. Every system of neighborhoods satisfying the conditions of Theorem 2.21 and inducing, in accordance with Theorem 2.21, the very same topology which was originally given in R is called a *system of neighborhoods of the topological space R*.

It follows from 2.24 that the elements of a system of neighborhoods of R constitute a basis in R; conversely, if each element Γ of a basis $\mathfrak{B}$ is taken to be a neighborhood of every point $x \in \Gamma$, the basis forms a system of neighborhoods of the space R.

EXAMPLE. The real line with its usual topology (as in analysis) has, among others, the following neighborhood systems: 1. A neighborhood of a point p is any open interval containing p. 2. A neighborhood of a point p is any open interval with rational endpoints containing p. 3. A neighborhood of p is any interval of the form $(p - r, p + r)$, where r is any positive rational number.

DEFINITION 2.251. A topological space is said to have a *countable basis* if it has a basis consisting of a countable set of elements. Such a space is also said to be a space of "*countable weight.*" In general, the least cardinal number τ such that a topological space has a basis of power τ is called the *weight* of the space.

EXAMPLE. The set of all open intervals with rational endpoints is a countable basis for the real line.

2.26. Let $\mathfrak{B}$ be a basis for the space R and let $M \subset R$. To obtain a basis for M, it suffices to take the sets which are the intersections of M with the elements of $\mathfrak{B}$. Hence the weight of M does not exceed the weight of R.

§2.3. Metric and metrizable spaces.

DEFINITION 2.3. *A set of arbitrary elements (points of the space) in which every two elements x and y are assigned a nonnegative number $\rho(x, y)$, the distance between x and y, is called a metric space if the following conditions are satisfied*:

1. $\rho(x, y) = 0$ *if, and only if, x and y are identical (axiom of identity).*
2. $\rho(x, y) = \rho(y, x)$ *(axiom of symmetry).*
3. *If x, y, z are any three points of the space,*

$$\rho(x, y) + \rho(y, z) \geq \rho(x, z)$$

(triangle axiom).

A nonnegative function $\rho(x, y)$ of two variable points x, y of the space R satisfying these conditions is called a *metric* of the metric space.

REMARK. Every subset of a metric space R is itself a metric space with metric the same as that in R.

A metric of a metric space defines a topology in the space in the following way. Let us call the greatest lower bound of the nonnegative numbers $\rho(p, x)$, $x \in M$, the *distance between the point p and the set M*. Define the closure of a set M in the metric space R to be the set of all points p for which $\rho(p, M) = 0$. Finally, in accord with Remark 3 of 2.1, call the set M *closed* if it coincides with its own closure [or, M is closed if it contains every point p such that $\rho(p, M) = 0$]. It is easily verified that

this topology satisfies the axioms of a topological space. We shall say that the topology just defined in a metric space is the *topology induced by the metric* or the *natural topology of the metric space.*

2.31. Let R be a metric space. The least upper bound $d \leq \infty$ of the numbers $\rho(x, y)$, $x, y \in R$, is called the *diameter* of R.

2.32. Let $\epsilon > 0$. We shall call the set $S(P, \epsilon)$ of all points $x \in R$ such that $\rho(P, x) < \epsilon$ an *ϵ-neighborhood* of the set (or point) P of the metric space R.

REMARK. An ϵ-neighborhood is also called a *spherical neighborhood* (of radius ϵ). It is easily seen that spherical neighborhoods are open sets.

2.33. Let M be a subset of a metric space R; then $p \in \bar{M}$ if, and only if, every spherical neighborhood $S(p, \epsilon)$ of p contains at least one point of M.

2.331. The spherical neighborhoods of the points of a metric space R form a neighborhood system in R (in the sense of Def. 2.25).

2.34. A sequence

$$x_1, x_2, \cdots, x_n, \cdots \tag{2.34}$$

of points of a metric space R *converges*, by definition, to a point $x \in R$ if $\lim_{n\to\infty} \rho(x, x_n) = 0$, i.e., if every (spherical) neighborhood of x contains all but a finite number of the points of the sequence.

2.35. A point p of a metric space R is a contact point of a set M if, and only if, there is a sequence of points of M which converges to p. A point p is an accumulation point of M if, and only if, M contains a sequence of distinct points of M converging to p.

2.36. A metric space R has a countable basis if, and only if, it contains a countable set dense in R.

Indeed, if $U_1, \cdots, U_n, \cdots$ is a basis for the metric space R and $x_n \in U_n$, then the set of all x_n is dense in R. Conversely, if $D = \{x_n\}$ is dense in R, then the set of all $S(x_n, r)$, where $x_n \in D$ and r is any positive rational number, is a basis for R.

It follows from 2.36 and 2.26 that:

2.37. If a metric space R contains a denumerable set dense in R and $A \subset R$ is infinite, then A also contains a denumerable dense set.

EXAMPLES OF METRIC SPACES. 1. *The Euclidean n-space R^n* is the space of all sequences of n real numbers $x = (x_1, \cdots, x_n)$, with the distance between two points $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ given by

$$\rho(x, y) = [(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2]^{\frac{1}{2}}$$

2. Every subset of a Euclidean space is also a metric space. Among these subsets we note particularly:

the *closed solid n-sphere* $\bar{E}^n$ with center a and radius ρ which consists of all points $x \in R^n$ at a distance $\leq \rho$ from the point a;

the *open solid n-sphere* E^n with center a and radius ρ consisting of all points $x \in R^n$ at a distance $<\rho$ from the point a;

the $(n-1)$-*sphere* $S^{n-1} = \bar{E}^n \setminus E^n$ with center a and radius ρ consisting of all points $x \in R^n$ whose distance from a is equal to ρ.

3. The *Hilbert space* R^∞ has as points the set of all infinite sequences of real numbers

$$x = (x_1, \cdots, x_n, \cdots)$$

such that $\sum_{n=1}^{\infty} x_n^2 < \infty$; the distance between two points $x = (x_1, \cdots, x_n, \cdots)$ and $y = (y_1, \cdots, y_n, \cdots)$ is given by the formula

$$\rho(x, y) = [\sum_{n=1}^{\infty} (x_n - y_n)^2]^{\frac{1}{2}}.$$

This is an immediate generalization of the distance formula in Euclidean space [3].

4. The set of all points $x = (x_1, \cdots, x_n, \cdots)$ of Hilbert space such that $|x_n| \leq (\frac{1}{2})^n$, $n = 1, 2, \cdots$, is called the *Hilbert parallelotope* (in analogy with the ordinary parallelotope whose width is half its length and whose thickness is half its width).

REMARK. (Urysohn's theorem.) *Every metric space which contains a countable dense subset is homeomorphic* (see 2.45) *to a subset of the Hilbert parallelotope* [4].

§**2.4. A continuous mapping** of a topological space X into a topological space Y may be defined in each of the following equivalent ways:

2.41_1. The mapping C is *continuous* if the inverse image $C^{-1}(B)$ of every set B closed in Y is closed in X.

2.41_2. The mapping C is *continuous* if the inverse image of every set open in Y is open in X.

2.41_3. The mapping C is *continuous* if $C(\bar{A}) \subseteq \overline{C(A)}$ for every subset $A \subseteq X$.

2.42. (Cauchy's definition of continuity.) A mapping C of a topological space X into a topological space Y is *continuous at a point* $a \in X$ if for every neighborhood $V(b)$ of the point $b = C(a) \in Y$ there exists a neighborhood $V(a)$ of the point a such that $C[V(a)] \subseteq V(b)$.

2.43. A mapping C is continuous in the sense of Defs. 2.41_1–2.41_3 if, and only if, it is continuous at every point of the space X in the sense of Def. 2.42.

Moreover, in the case of metric spaces, we have

2.44. A mapping C of a metric space X into a metric space Y is continuous if, and only if, the convergence of an arbitrary sequence

$$x_1, \cdots, x_n, \cdots \text{ in } X$$

to a point $x \in X$ always implies the convergence of the sequence

$$C(x_1), C(x_2), \cdots, C(x_n), \cdots \text{ in } Y$$

to the point $C(x) \in Y$.

2.45. Let C be a continuous (1–1) mapping of a topological space X onto a topological space Y. If the mapping C^{-1} of Y onto X, the inverse of C, is continuous, the mapping C is said to be *bicontinuous* or *topological*; topological mappings are also known as *homeomorphisms*. Two topological spaces are *homeomorphic* if one of them can be mapped topologically onto the other.

A topological space which is homeomorphic to a metric space is called a *metrizable* space.

§2.5. Uniform convergence of mappings.

DEFINITION 2.51. A sequence

$$C_1, C_2, \cdots, C_n, \cdots$$

of mappings of a set X into a metric space Y is *uniformly convergent* to the mapping C of X into Y if for every $\epsilon > 0$ there exists a natural number $n(\epsilon)$ such that

$$\rho[C(x), C_n(x)] < \epsilon$$

for $n > n(\epsilon)$ and all $x \in X$. The proof of the following theorem is the same as that given in books on analysis:

2.52. The limit of a uniformly convergent sequence of continuous mappings of a topological space X into a metric space Y is a continuous mapping of X into Y.

§2.6. Topological product of spaces. Let us apply 2.2 to define the *topological product* of topological spaces.

The product of two sets is, after Cantor, the collection of all pairs (x, y), $x \in X$, $y \in Y$. The product of the sets X_α of a system $\mathfrak{X} = \{X_\alpha\}$ is the set of all systems of the form

$$\xi = \{x_\alpha\}$$

containing a single element x_α of each set X_α. The points of the topological product of the topological spaces X_α are, by definition, the elements of the product of the sets X_α. The points $x_\alpha \in X_\alpha$ are referred to as the "coordinates" of the point $\xi = \{x_\alpha\}$. We shall define a topology first for the product of two spaces X and Y: a neighborhood of a point $\xi = (x, y)$ is the product of arbitrary neighborhoods of the points x and y in X and Y, respectively. It is not difficult to prove that this topology satisfies conditions 1_V–2_V of Theorem 2.21.

A generalization of this topology to an arbitrary number of spaces X_α was first found by A. N. Tihonov: to obtain a neighborhood of a point $\xi^0 = \{x^0{}_\alpha\}$, first choose any finite number of coordinates $x^0{}_\alpha$ of this point, say $x^0{}_{\alpha_i}$ $(1 \leq i \leq s)$, and then choose neighborhoods $V(x^0{}_{\alpha_i}) \subseteq X_{\alpha_i}$ of each of these coordinates. A neighborhood of the point ξ^0 consists, by definition, of all $\xi = \{x_\alpha\}$ such that $x_{\alpha_i} \in V(x_{\alpha_i})$ $(1 \leq i \leq s)$, with the remaining coordinates assuming all possible values. We may again show that the conditions of Theorem 2.21 are satisfied.

EXAMPLES. 1. The plane is the topological product of two lines, the torus [the surface obtained by rotating a circumference about an axis lying in the plane of the circumference and not intersecting it (Fig. 1)] is the topological product of two 1-spheres, three-dimensional space is the product of three lines. In general, the Euclidean n-space (with the topology defined by its usual metric) is the topological product of n lines.

The topological product of n 1-spheres is known as an n-dimensional torus.

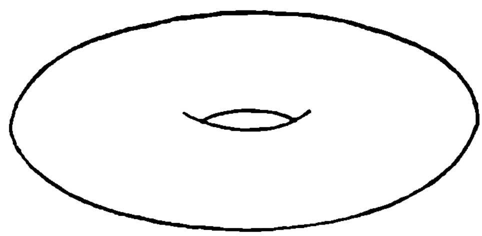


FIG. 1

2. Let $oxyz$ be a system of coordinates in three-dimensional space. In the plane $z = 0$ consider the unit circumference $S^1 \equiv x^2 + y^2 = 1$ and a mechanism M_1 consisting of two rods oa and ac, each of unit length and fastened at the point a by a spherical hinge in such a way that the rod oa can rotate freely in the plane $z = 0$ around the fixed point o, while the rod ac is free to rotate in the three-dimensional space around the point a.

Let (oa, ac) be a definite position of the system of two rods. As a neighborhood of this position it is natural to take the set of all positions $(oa', a'c')$ for which the distances $\rho(a, a')$, $\rho(c, c')$ are less than a given $\epsilon > 0$. This converts the set of all positions of the mechanism into a topological space P.

We shall prove that the space P is homeomorphic to the topological product of the 1-sphere S^1 and a 2-sphere S^2. Indeed, a definite position of the system of two rods oa and ac determines the point a of the 1-sphere $S^1 \equiv x^2 + y^2 = 1$ and the point of the 2-sphere $S^2 \equiv x^2 + y^2 + z^2 = 1$ in which this sphere intersects the ray emanating from the point o in the direction of the vector ac. Conversely, every pair of points, $p \in S^1$, $p' \in S^2$ determines a position op of the first rod and also a position of the second

rod, obtained by laying off from the point p a vector equal to the vector op'. This defines a (1–1) correspondence between the space P and the topological product of a 1-sphere and a 2-sphere. It is easy to see that this correspondence is bicontinuous. We can say, briefly, that the *manifold* (see 5.3) *of all positions of the mechanism* M_1 *is the topological product of the* 1-*sphere* S^1 *and the* 2-*sphere* S^2.

3. Let us now consider a mechanism M_2 consisting of a rod oa of unit length, which can rotate freely in space around the point o (the origin of coordinates), and a rod ac (also of unit length) fastened to the rod oa at the point a in such a way that it can rotate freely in a plane passing through the point a and perpendicular to the line oa, i.e., in the plane tangent to the sphere S^2 at the point a.

The set of positions of this mechanism may again be converted into a topological space Q in the following natural way: as a neighborhood of a position (oa, ac) take all positions $(oa', a'c')$ for which the distances $\rho(a', a)$ and $\rho(c', c)$ are less than a given $\epsilon > 0$. The reasoning which established the topological equivalence of the space P (Example 2) with the topological product of a 1-sphere and a 2-sphere is not applicable to this case (the reader should verify this immediately). *The space Q is not homeomorphic to the topological product of a* 1-*sphere and a* 2-*sphere.*

We shall not prove this assertion in full, but will show that Q, in any case, cannot be decomposed into the product of a 1-sphere and a 2-sphere in so natural a way as in the case of P. To each point of the space Q, i.e., to each position (oa, ac) of the mechanism, there corresponds a definite point of the sphere S^2. We shall prove that there does not exist a homeomorphism between Q and the topological product $S^1 \times S^2$ of S^1 and S^2 satisfying the following condition: to each point $q = (oa, ac)$ of Q there corresponds a point $(p, a) \in S^1 \times S^2$, where p is any point of S^1 and the point a is determined by the condition $q = (oa, ac)$.

Let us suppose that such a homeomorphism exists and consider the set Φ of all points $(p_0, p') \in S^1 \times S^2$, where p_0 is a fixed point of S^1 and p' takes all values on S^2. Our assumptions imply that a definite vector $V(p')$ of unit length emanating from the point p' of S^2 and lying in the plane tangent to this sphere at the point p' corresponds to each $(p_0, p') \in \Phi \subseteq S^1 \times S^2$. Hence the vector $V(p')$ is a continuous function of p'. We have therefore obtained on the sphere S^2 a continuous field of tangent vectors which vanish nowhere. We shall prove in Chapter XVI, however, that such continuous vector fields do not exist on the 2-sphere (XVI, 5.510).

4. Consider a mechanical system consisting of a point moving on a circumference with arbitrary speed. Each state of this system is determined by two data: the position of the point on the circumference and its speed. Topologizing this collection of states in a natural way (proximity of posi-

tions and proximity of speeds), we obtain as the manifold of states ("phase space") the product of a 1-sphere and a line, i.e., an infinite cylinder.

5. Let us call a space consisting of two isolated points a *doublet*. The product of a denumerable number of doublets is homeomorphic to the Cantor perfect set. If τ is any cardinal number, we shall denote by D_τ (dyadic discontinuum) the very remarkable space which is the product of τ doublets.

The product of τ spaces each of which is homeomorphic to a closed segment of the real line is also very important; it is denoted by R_τ. The space $R_{\aleph_0}$, i.e., the topological product of a denumerable number of closed segments, is homeomorphic to the Hilbert parallelotope (see 2.3, Example 4). The reader should prove this assertion.

6. Let R be a space consisting of two points a, b with the following topology: there are exactly three closed sets in R, the empty set, the set consisting of the point a, and the set consisting of both points a, b. We shall call this space a *connected doublet* (see 3.1 for a definition of connectedness).

Let τ be an arbitrary cardinal number and denote by F_τ the topological product of τ connected doublets. The properties of the space F_τ are interesting even for finite τ.

§3. Connectedness

§3.1. Definition and basic theorems.

DEFINITION 3.1. A topological space is said to be *connected* if it is not the union of two disjoint nonempty closed sets.

Since every subset of a topological space is itself a topological space (see 2.16), this definition of connectedness applies also to subsets of topological spaces.

EXAMPLE 1. The real line is connected, as are open and closed segments [5].

We shall prove several theorems on connectedness.

FUNDAMENTAL LEMMA 3.11. *If A and B are two disjoint closed subsets of a topological space R and a set $M \subseteq A \cup B$ is connected, then either $M \subseteq A$ or $M \subseteq B$.*

Indeed, in the contrary case, $M = (M \cap A) \cup (M \cap B)$, where $M \cap A$ and $M \cap B$ are disjoint nonempty sets closed in M.

3.12. *If every two points of a space R are contained in a connected set $M \subseteq R$, then R is connected.*

In fact, let

$$R = A_1 \cup A_2,$$

where A_1 and A_2 are disjoint nonempty closed sets. Let $p_1 \in A_1$, $p_2 \in A_2$, and let M be a connected subset of R containing p_1 and p_2.

By virtue of 3.11, the set M is contained in one of the sets A_1, A_2. Then both points p_1, p_2 are contained in one of the sets A_1, A_2, a contradiction.

REMARK 1. The connectedness of a segment and 3.12 imply that every convex set is connected. In particular, the space R^n is connected.

3.13. *The union of two connected subsets A and B of a topological space R such that $A \cap B \neq 0$ is connected.*

For, let $A \cup B = C$.

If

$$C = C_1 \cup C_2,$$

where C_1, C_2 are disjoint sets closed in C, then, by 3.11, each of the sets A, B is contained in one of the two sets C_1, C_2, say $A \subseteq C_1$. Then $A \cap B \subseteq C_1$, whence $B \subseteq C_1$, i.e., C_2 is empty, q.e.d.

DEFINITION 3.14. A finite sequence of sets

$$A_1, \cdots, A_s \tag{3.14}$$

is called a *chain of sets* (more precisely: a chain connecting A_1 to A_s) if $A_j \cap A_{j+1} \neq 0$ $(1 \leq j \leq s - 1)$.

REMARK 2. The chain (3.14) is said to be *closed* if, in addition, $A_s \cap A_1 \neq 0$.

A system (of any power) of sets is said to be *chained* if any two sets of the system can be connected by a chain made up of elements of this system.

It follows from 3.13 that:

3.14 The union of a chain of connected sets of a topological space R is connected.

Furthermore, 3.14 and 3.12 imply

3.15. *Let A_α be a chained system of connected sets of a topological space R. Then the union of the sets A_α is connected.*

In particular, we have

3.16. *The union of any number of connected sets of a topological space R, each pair of which has a nonempty intersection, is connected.*

DEFINITION 3.17. An open connected set of a topological space R is called a *domain of R*.

3.18. *Let $\mathfrak{G}$ be a system of nonempty domains Γ_α in a topological space R. The union Γ of the domains $\Gamma_\alpha \in \mathfrak{G}$ is connected only if $\mathfrak{G}$ is a chained system.*

Proof. Suppose that the system $\mathfrak{G}$ is not chained. Then there exist two domains Γ_1, $\Gamma_2 \in \mathfrak{G}$ which cannot be connected by any chain of elements of the system $\mathfrak{G}$. Denote by Γ' the union of all the elements of the system

$\mathfrak{G}$ which can be connected to Γ_1 by chains of elements of $\mathfrak{G}$ and denote by Γ'' the union of all the remaining elements of the system $\mathfrak{G}$. The sets Γ' and Γ'' are disjoint nonempty open sets and their union is Γ. It follows that Γ is not connected. This proves 3.18.

3.19. *If $A \subseteq R$ is connected and $A \subseteq B \subseteq \bar{A}$, then B is also connected.*

Proof. Let

$$B = B_1 \cup B_2,$$

where B_1 and B_2 are disjoint and closed in B. We shall show that one of the sets B_1, B_2 is empty. Since A is connected, A is contained in one of the sets B_1, B_2, say B_1. But B_1 is closed in B. Hence the set of all contact points of A contained in B, i.e., $\bar{A} \cap B = B$, is contained in B_1. Therefore, B_2 is empty, which was to be proved.

EXAMPLE 2. Let the set A consist of all the points of the curve $y = \sin 1/x$, $0 < x < 1$ (Fig. 2). Since this curve is homeomorphic to the real line, A is connected. If any set of points lying on the segment $-1 \leq y \leq 1$ of the axis of ordinates (e.g., the two points $y = -\frac{1}{2}, y = \frac{1}{2}$) is added to A, the result is a connected set B.

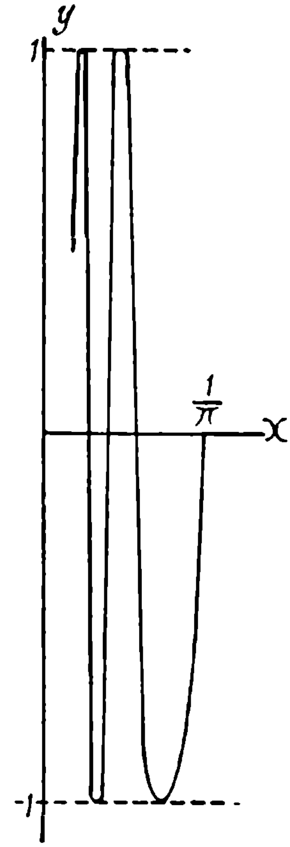


FIG. 2

§3.2. **Components.** Let p be a point of a topological space R. Since a set consisting of one point is connected, there is a connected set in R containing p. The union of all connected sets containing p is connected (see 3.16) and is the largest connected set in R containing p; it is called the *component* of the point p in R. It follows from 3.19, that the component of each point p of a topological space R is a closed set. 3.13 implies that the components of two points p and p' are either identical or disjoint.

Therefore,

3.21. *Every topological space can be partitioned into the components of its points.*

EXAMPLE 3. Let P be the Cantor perfect set on the segment $0 \leq x \leq 1$ of the axis of abscissas, and let Q consist of all points (x, y) of the plane such that $x \in P$, $0 \leq y \leq 1$. The sets Q_x, each of which consists of the points $(x, y) \in Q$ with x fixed and $0 \leq y \leq 1$, are components of the set Q. The set of these components has the power of the continuum.

EXERCISE. Prove that a continuous image of a connected topological space is a connected topological space.

§4. Separation axioms. Compactness [6]

§4.1. Separation axioms. A set $M \subset R$ is said to be *degenerate* if it consists of just one point.

4.1. Every open set containing a set M is called a *neighborhood of the set M.*

4.11 A topological space R is said to be a *T_0-space* if every two distinct degenerate subsets of R have distinct closures in R (see the examples at the end of this article).

It is easy to see that this definition is equivalent to the following:

4.11′. A space R is called a T_0-space if, given any two distinct points of R, *at least one* of the points has a neighborhood not containing the other point.

4.12. A topological space is called a *T_1-space* if all its degenerate sets are closed.

An equivalent definition is

4.12′. A space R is a T_1-space if, given any two distinct points of R, *each of the points* has a neighborhood not containing the other point.

4.13. A topological space is called a *T_2-space*, or a *Hausdorff space*, if every two distinct points have disjoint neighborhoods.

THEOREM 4.131. Let R be a T_0-, a T_1-, or a T_2-space, respectively. Let $\mathfrak{A}$ be a system of neighborhoods of R. Then the neighborhoods mentioned in 4.11′, 4.12′, and 4.13, respectively, can be chosen from among the neighborhoods of the system $\mathfrak{A}$.

4.14. A Hausdorff space is said to be *normal* if any two of its disjoint closed sets have disjoint neighborhoods.

This definition may also be given the following form:

4.141. A Hausdorff space R is *normal* if, for every closed set $A \subset R$ and neighborhood $V(A)$ of A, there exists a neighborhood $V_1(A)$ of A such that $\bar{V}_1(A) \subseteq V(A)$.

4.15. *Every metric space is normal* [7].

4.16 (Urysohn). *A space with a countable basis is metrizable if, and only if, it is normal* [8].

The proof of Theorem 4.16 depends on Theorem 4.21.

EXAMPLES. 1. The space R consists of two points. The only sets closed in R are the empty set and R. R is not a T_0-space.

2. A connected doublet (2.6, Example 6) is a T_0-space, but not a T_1-space. The space F_τ (2.6, Example 6) is also a T_0-space, but not a T_1-space, for every τ. The space F_τ has the following universal property: *it contains the topological image of every T_0-space of weight $\leq \tau$* (see Def. 2.251 and [9]). Conversely, every subset of the space F_τ is a T_0-space of weight $\leq \tau$. Finite T_0-spaces are the most important special cases of so called *discrete*

spaces (see §6). The only discrete spaces which are interesting are those which are not T_1-spaces (since all finite T_1-spaces consist of isolated points).

3. Let us adjoin to the segment $0 \le x \le 1$ with its usual topology a new point ξ. As a neighborhood of the point ξ take any set consisting of the point ξ and all except an arbitrary finite number of points of the segment $0 \le x \le 1$. The resulting topological space is a T_1-space, but not a T_2-space.

4. The spaces D_τ and R_τ, $\tau > \aleph_0$, are examples of nonmetric normal spaces. Every normal space of weight $\le \tau$ is homeomorphic to a subset of the space R_τ (Theorem of A. N. Tihonov [10]). All the subspaces of the space R_τ are T_2-spaces of weight $\le \tau$, but not all of them are normal.

§4.2. Theorems on continuous functions in normal spaces [11].

4.21 (Urysohn). *In order that a Hausdorff space R be normal, it is necessary and sufficient that, for every two disjoint nonempty closed subsets A_0 and A_1 of R, there exist a real function $f(x)$ continuous on all of R, such that $0 \le f(x) \le 1$ for all $x \in R$ and such that $f(x) = 0$* for $x \in A_0$ and $f(x) = 1$ for $x \in A_1$ [12].

4.22 (Brouwer-Urysohn). *Every continuous function $f(x)$ defined on a closed set A of a normal space R can be extended to all of R, i.e., a real function $F(x)$ can be constructed which is continuous on all of R and coincides with $f(x)$ on A* [13].

Corollary. Since a continuous mapping of a space R into Euclidean n-space R^n defines a system of n real functions $x_i = f_i(x)$, $x \in R$ (where $x_1, \cdots, x_n$ are the coordinates in R^n), 4.22 implies

4.23. Every continuous mapping f of a closed set A of a normal space R into R^n can be extended to all of R.

§4.3. Compactness [14].

Definition 4.31. A topological space R is said to be *compact* if every open covering of R contains a finite open covering of R.

A compact Hausdorff space is called a *bicompactum*. A metrizable compact space is called a *compactum*. A compactum is obviously a special case of a bicompactum.

4.32. *A metrizable space is a compactum if, and only if, every sequence of points of the space contains a convergent subsequence* [15].

4.33. *Among the subsets of Euclidean spaces the compacta are characterized as those which are closed and bounded* [16].

4.34. *Every closed subset of a bicompactum (compactum) is itself a bicompactum (compactum)* [17].

4.35. *If a bicompactum Φ is a subset of a Hausdorff space R, then Φ is a closed subset of R* [18].

§4.4. Further theorems on bicompacta. Metrization and imbedding theorems.

4.41. *Every bicompactum is a normal space* [19].

4.42 (Urysohn). *In order that a bicompactum be a compactum, i.e., be metrizable, it is necessary and sufficient that it have a countable basis* [20].

4.43 (Tihonov). *The topological product of an arbitrary finite or infinite number of bicompacta is a bicompactum* [21].

4.431. *The topological product of a countable number of compacta is a compactum* [22].

4.44 (Aleksandrov). *Among the Hausdorff spaces the compacta are characterized as those which are continuous images of the Cantor perfect set* [23].

4.45 (Aleksandrov). *Among the Hausdorff spaces the bicompacta of weight* $\leq \tau$ *are characterized as those which are continuous images of closed subsets of the space* D_τ [24].

REMARK. This result cannot be improved: for every noncountable τ there is a bicompactum of weight τ which is not a continuous image of the whole space D_τ (an example of such a bicompactum is the set of all ordinals $\leq \omega_\tau$, where a neighborhood of a given ordinal is any interval of ordinals containing it and ω_τ is the least ordinal of power τ).

§4.5. Continuous mappings of bicompacta.

4.51. *Every Hausdorff space which is a continuous image of a bicompactum is a bicompactum* [25].

4.511. *Every metric space which is a continuous image of a bicompactum (in particular, of a compactum) is a compactum* [26].

REMARK. It is somewhat harder to prove that every Hausdorff space which is a continuous image of a compactum is a compactum [27].

Consequently,

4.52. *A* (1–1) *continuous mapping of a bicompactum* X *onto a Hausdorff space* Y *is a topological mapping.*

4.53. *Every continuous mapping of a compactum* X *onto a compactum* Y *is uniformly continuous* [28], *i.e., for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\rho[C(x), C(x')] < \epsilon$ *for* $x \in X$, $x' \in X$, *and* $\rho(x, x') < \delta$.

§5. Upper semi-continuous decompositions of compacta and their relation to continuous mappings (identifications). Locally compact spaces. Topological manifolds. Examples

§5.1. Upper semi-continuous decompositions. The space of a given decomposition [29]. Every continuous mapping C of a compactum X onto a compactum Y induces a decomposition $\mathfrak{C}$ of X into mutually disjoint closed sets, the inverse images $C^{-1}(y)$ of the points $y \in Y$.

This decomposition has the following property:

5.11. *If A_0 is an element of the decomposition $\mathfrak{C}$ and Γ is an arbitrary neighborhood of the set A_0 in the space X, there exists a neighborhood $\Gamma_1 \subset \Gamma$ of the set A_0 in X such that every set $A \in \mathfrak{C}$ which intersects Γ_1 is contained in Γ.*

Indeed, let $A_0 = C^{-1}(y_0)$ and suppose that condition 5.11 is not satisfied (we shall regard the compactum X as defined by a metric). Then there is a sequence of sets

$$A_n = C^{-1}(y_n) \in \mathfrak{C}$$

with the following properties:

1. There is a point $x'_n \in A_n$ whose distance from $A_0 < 1/n$.

2. There is a point $x''_n \in A_n$ whose distance from A_0 is greater than some fixed $\epsilon > 0$. Without loss of generality we may suppose that both sequences $\{x'_n\}$ and $\{x''_n\}$ converge:

$$\lim x'_n = x' \in A_0,$$

$$\lim x''_n = x'' \in X \setminus A_0,$$

where $x'' \in X \setminus A_0$ implies that $C(x'') \neq y_0$. Since the mapping C is continuous, $y_n = C(x'_n)$ converges to $C(x') = C(A_0) = y_0$. On the other hand, since $x''_n \in A_n$, $C(x''_n) = y_n$ and, by the continuity of C,

$$\lim y_n = \lim C(x''_n) = C(x'') \neq y_0 .$$

This contradiction proves the assertion.

DEFINITION 5.12. *A decomposition of a compactum X into mutually disjoint closed sets A is said to be upper semi-continuous if it has property* 5.11.

5.13. Every decomposition $\mathfrak{C}$ of a compactum X into mutually disjoint closed sets A induces a topological space Y, the *decomposition space,* in the following way: the sets A are, by definition, the points of the space Y; to define a neighborhood of a point $A_0 \in Y$, choose any neighborhood Γ of the set A_0 in the space X and let the corresponding neighborhood $U_\Gamma(A_0)$ of the point A_0 in the space Y consist of all $A \in Y$ for which $A \subseteq \Gamma$ in X.

It is easy to verify that the resulting neighborhoods in Y satisfy the conditions of Theorem 2.21. Hence Y is a topological space and, since X is normal, Y is a Hausdorff space.

Furthermore,

5.14. *If the decomposition $\mathfrak{C}$ is upper semi-continuous, the assignment to each point $x \in X$ of the set $A \in Y$ containing x is a continuous mapping C of X into Y.*

In fact, let $x \in A_0 \in Y$; then $C(x) = A_0$. Let $U_\Gamma(A_0)$ be any neighborhood of the point $A_0 \in Y$. By 5.11, the neighborhood Γ of the set A_0 in X which generates the neighborhood $U_\Gamma(A_0)$ contains a neighborhood

Γ_1 of A_0 in X with the property that every $A \in \mathfrak{C}$ which intersects Γ_1 is contained in Γ. Then Γ_1 is a neighborhood of the point x in X and C maps Γ_1 into the given neighborhood $U_\Gamma(A_0)$ of the point $A_0 = C(x) \in Y$. This proves the continuity of the mapping (Cauchy's criterion 2.42).

Since a Hausdorff space which is a continuous image of a compactum is itself a compactum (see 4.5, Remark), it follows that:

5.15. The space induced by an upper semi-continuous decomposition of a compactum X is a compactum, since it is a continuous image of the compactum X.

REMARK. If a compactum Y is the space of the upper semi-continuous decomposition $\mathfrak{C}$ of a compactum X or is homeomorphic to this space, we shall say that the compactum Y is the result of *identification* of certain points of X; namely, all the points of a set $A \in \mathfrak{C}$ are identified and so give rise to a point of the space Y. If every A is a finite set, this identification may be thought of as effected by "pasting together" the points involved.

§5.2. Examples of upper semi-continuous decompositions and identifications. The projective n-space.

1. Let X be a 2-sphere with a system of geographic coordinates defined on it. Let the elements of the upper semi-continuous decomposition be: a) the pair of poles of the given system of coordinates, and b) the circles of latitude of this system. The space of this decomposition is clearly homeomorphic to a 1-sphere.

2. Let X be a torus and consider the upper semi-continuous decomposition of the torus into its meridians $\psi = c$ (Fig. 3; the meridian $\psi = 0$ is drawn as a solid curve). The space of this decomposition is also homeomorphic to a 1-sphere.

3. In general, if Z is the topological product of compacta X and Y, the sets A_y consisting of all points $(x, y) \in Z$, where $y \in Y$ is fixed and x runs over the whole space X, are the elements of an upper semi-continuous

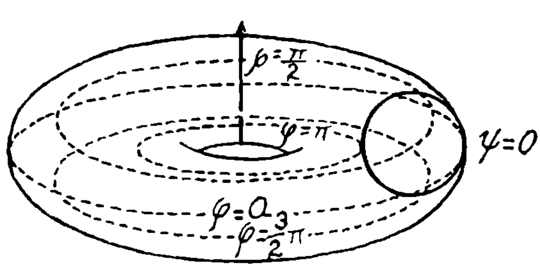


FIG. 3

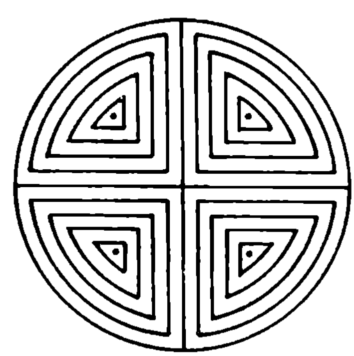


FIG. 4

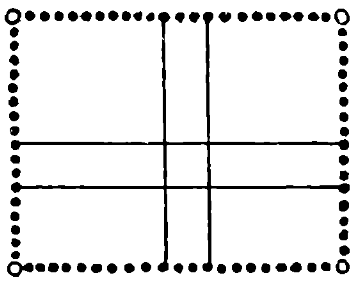


FIG. 5

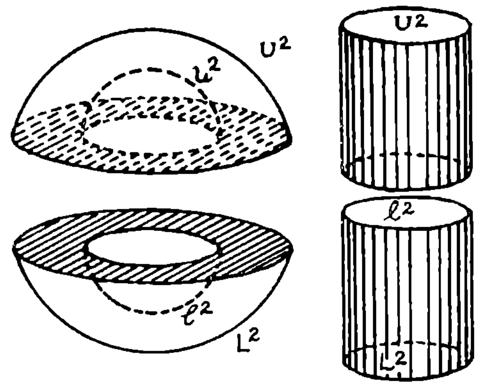


FIG. 6

decomposition of the space Z; the space of this decomposition is homeomorphic to Y.

4. Let us denote by $\mathfrak{C}$ the upper semi-continuous decomposition of a circle represented in Fig. 4 (the circumference together with both mutually perpendicular diameters form one element of the decomposition). The space of this decomposition is homeomorphic to a compactum consisting of four closed straight line segments with one common endpoint.

5. Let X be the square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the plane (x, y). Consider the decomposition $\mathfrak{C}$ of the square X whose elements are: a) the individual interior points of the square; b) the pairs of points $\{(x, 0), (x, 1)\}$, $0 < x < 1$; c) the pairs of points $\{(0, y), (1, y)\}$, $0 < y < 1$; d) the quadruple of points $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The space of this decomposition is homeomorphic to a torus. In brief, *the torus is the result of identifying corresponding points on opposite sides of the square* (Fig. 5).

6. In an analogous sense, identification of corresponding points of each pair of opposite faces of a solid cube yields the *three-dimensional torus* (topological product of three 1-spheres).

7. Let Q^3 be the closure of the domain of three-dimensional space contained between two concentric spheres S^2 and s^2, and identify corresponding points of both spheres (i.e., the points of intersection of a ray emanating from the center with the two spheres). The result is the *topological product of a 2-sphere and a 1-sphere.* [Let the 2-sphere be one of the concentric spheres. The segment joining a point of this sphere to the corresponding point of the other sphere becomes a 1-sphere after identification of its endpoints (Trans.).]

This identification can be represented in still another way (Fig. 6).

Let us cut the solid Q^3 with an equatorial plane into two congruent solids $Q^3{}_u$ and $Q^3{}_l$. This divides each of the spheres S^2 and s^2 into two hemispheres: upper U^2 and u^2, and lower L^2 and l^2. Denote the equatorial circumferences of the spheres S^2 and s^2 by S' and s', respectively.

The solid Q^3_u can be thought of as a deformed solid cylinder with bases U^2 and u^2 and lateral surface $\Pi^2 = \Pi^2_u$, the latter in the form of a plane ring which is hatched in the sketch.

The solid Q^3_l is also a cylinder with lateral surface $\Pi^2 = \Pi^2_l$ and bases L^2 and l^2.

In each of the solids Q^3_u and Q^3_l corresponding points of the upper and lower bases are to be identified; this identification converts both solids into two three-dimensional rings (a three-dimensional ring is a bounded solid whose boundary is a torus). Finally, it remains only to do away with the equatorial cut, i.e., to identify corresponding points of both toruses Π^2_u and Π^2_l.

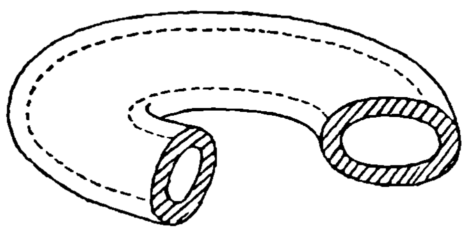


FIG. 7

Hence the topological product of a 2-sphere and a 1-sphere is obtained by "doubling" a three-dimensional ring, i.e., by identifying corresponding points of the boundaries of two congruent three-dimensional rings.

REMARK. It is easily and immediately verified that the space resulting from the doubling of a three-dimensional ring is the topological product of a 2-sphere and a 1-sphere. [For the 1-sphere take a circle of latitude of one of the toruses. At each point of this circle the meridian circle passing through the point is identified with the corresponding meridian of the other torus, but the interiors of the meridian circles are not identified. This gives a 2-sphere at each point of the 1-sphere (Trans.).]

EXERCISE. Prove that a three-dimensional torus is obtained from the domain included between two coaxial toruses (Fig. 7) by identifying corresponding points of the two bounding toruses.

8. Projective space. Let $x_0, \cdots, x_n$; $y_0, \cdots, y_n$ be two sequences of $n + 1$ real numbers, each of which contains at least one number different from zero. The two sequences are said to be *proportional* or *to define the same ratio* if $x_i y_k = x_k y_i$ $(0 \leq i, k \leq n)$. It is easy to see that the relation of proportionality of numerical sequences (consisting of the same number of elements) is reflexive, symmetric, and transitive. Therefore, the set of all sequences $x_0, \cdots, x_n$ (n fixed and at least one of the x_i different from zero) is partitioned into disjoint classes, each class consisting of proportional sequences. These classes are called *ratios*. Each ratio is a class of

proportional sequences. The ratio of the sequence $x_0, \cdots, x_n$ is denoted by $(x_0 : x_1 : \cdots : x_n)$. The sequence whose terms are all zero has no ratio. Given a ratio $(x_0 : x_1 : \cdots : x_n)$, it is always possible to choose a sequence $x_0, \cdots, x_n$ in the ratio such that $x_0^2 + \cdots + x_n^2 = 1$.

We may now define the projective n-space as follows: the points of the projective n-space are the ratios $x = (x_0 : \cdots : x_n)$ of all sequences of $n + 1$ real numbers $x_0, \cdots, x_n$ of which at least one is different from zero.

A neighborhood of a point $a = (a_0 : a_1 : \cdots : a_n)$ is defined by choosing in the ratio $(a_0 : a_1 : \cdots : a_n)$ a sequence $a_0, \cdots, a_n$ such that

$$a_0^2 + \cdots + a_n^2 = 1$$

and then taking all sequences $x_0, \cdots, x_n$ for which $x_0^2 + \cdots + x_n^2 = 1$ and $| a_i - x_i | < \epsilon$. The points (ratios) of the projective space corresponding to such sequences form, by definition, a neighborhood $S(a, \epsilon)$ of the point a. This definition permits the construction of several models of the projective space. Thus, for instance, each point $(x_0 : x_1 : \cdots : x_n) \in P^n$ can be made to correspond to the line of Euclidean $(n + 1)$-space R^{n+1} $(\xi_0 ; \cdots ; \xi_n)$ passing through the origin of coordinates and defined by the equations

$$(\xi_0/x_0) = (\xi_1/x_1) = \cdots = (\xi_n/x_n),$$

where the ξ_i are running coordinates.

It is left to the reader to prove that if the distance between two straight lines is defined as the angle φ between them, $0 \leq \varphi < \pi/2$, then the set of all these straight lines is converted into a metric space; and that the correspondence established above between this metric space and the space P^n is topological.

We have thus constructed a geometrical model of the projective space, in which the points are straight lines passing through the origin. Now let R^n be any n-plane in R^{n+1} which does not contain the origin and, as the representative of each straight line (passing through the origin), let us take its point of intersection with R^n. This yields the usual interpretation of the projective space as a Euclidean space closed by elements at infinity.

We obtain another model by taking as the representative of each straight line its two points of intersection with the unit sphere

$$\xi_0^2 + \xi_1^2 + \cdots + \xi_n^2 = 1$$

in R^{n+1}. This corresponds to the choice in each ratio $(x_0 : x_1 : \cdots : x_n)$ of a sequence $x_0, \cdots, x_n$ satisfying the condition $x_0^2 + \cdots + x_n^2 = 1$.

Hence it is clear that the n-sphere S^n, after identification of its diametrically opposite points, becomes the projective n-space.

Let us divide the sphere $\xi_0^2 + \cdots + \xi_n^2 = 1$ into two hemispheres $\xi_n > 0$

and $\xi_n < 0$ with the equatorial plane $\xi_n = 0$, and let us add the equator $\xi_n = 0$, $\xi_0^2 + \cdots + \xi_n^2 = 1$ to the upper hemisphere $\xi_n > 0$, denoting the resulting compactum $\xi_0^2 + \cdots + \xi_n^2 = 1$, $\xi_n \geq 0$ by Q^n.

Each pair of diametrically opposite points of S^n not lying on the equator S^{n-1} has its (unique) representative in Q^n, so that only diametrically opposite points of the equator need be identified. *To obtain projective n-space it, therefore, suffices to add to one hemisphere of the n-sphere S^n its equator S^{n-1} and to identify diametrically opposite points of the equator. Since the compactum Q^n is homeomorphic to a closed solid n-sphere, projective n-space is obtained from a closed solid n-sphere by identifying diametrically opposite points of its boundary.*

§5.3. **Locally compact spaces. Topological manifolds. Examples.** A topological space is said to be *locally compact* if every point of the space has a neighborhood whose closure is compact.

For instance, Euclidean n-space is locally compact (but, clearly, not compact).

A connected locally compact space with a countable basis, each point of which has a neighborhood homeomorphic to Euclidean n-space, is called an *n-dimensional topological manifold* or, briefly, an *n-manifold*. A manifold is said to be *closed* if it is compact; manifolds which are not closed are called *open*.

EXAMPLES OF CLOSED MANIFOLDS.

1. The n-sphere (see 2.3, Example 2).

2. The n-dimensional torus, i.e., the product of n 1-spheres; in general, *the topological product of an arbitrary finite number of closed manifolds is a closed manifold* (in particular, a topological product of n-spheres is a closed manifold).

3. [30]. Let us consider all sextuplets of real numbers

$$p_{12}\,,\ p_{13}\,,\ p_{14}\,,\ p_{34}\,,\ p_{42}\,,\ p_{23}$$

whose elements are not all zero and which satisfy

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0. \tag{5.31}$$

Each such sextuplet determines a line in projective 3-space [if, e.g., $p_{12} \neq 0$, the sextuplet determines the line passing through the points $(0: p_{12} : p_{13} : p_{14})$ and $(-p_{12} : 0: p_{23} : p_{24})$] whose Plücker coordinates are the six numbers p_{ij}; and two sextuplets determine the same line if, and only if, they are proportional to each other. [If a line passes through the points $(x_1 : x_2 : x_3 : x_4)$ and $(y_1 : y_2 : y_3 : y_4)$, its Plücker coordinates (determined to within proportionality) are $p_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$; hence the notation p_{ij}. Then the deter-

minant having first and third rows x_1, x_2, x_3, x_4 and second and fourth rows y_1, y_2, y_3, y_4 is clearly equal to zero; this is also expressed by (5.31).]

Conversely, every line determines a unique class of proportional sextuplets. Hence the lines of projective 3-space are in (1–1) correspondence with the ratios of sextuplets of real numbers which satisfy (5.31).

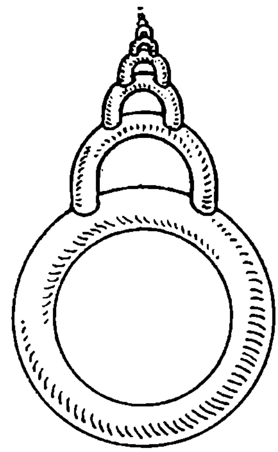


FIG. 8

Neighborhoods in the space of lines are defined by analogy with the neighborhoods already defined in the projective space. Let a line be determined by the ratio $(p^0_{12} : p^0_{13} : p^0_{14} : p^0_{34} : p^0_{42} : p^0_{23})$ and let p^0_{12}, p^0_{13}, p^0_{14}, p^0_{34}, p^0_{42}, p^0_{23} be a "normal" representative of this ratio, i.e., a sextuplet of the ratio for which $(p^0_{12})^2 + \cdots + (p^0_{23})^2 = 1$. Now let p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23} be all sextuplets such that $p_{12}^2 + \cdots + p_{23}^2 = 1$ and $| p^0_{12} - p_{12} | < \epsilon$, $\cdots$, $| p^0_{23} - p_{23} | < \epsilon$. The lines $(p_{12} : p_{13} : \cdots : p_{23})$ corresponding to these sextuplets generate an ϵ-neighborhood of the line $(p^0_{12} : \cdots : p^0_{23})$. This topology converts the set of all lines into a closed 4-manifold.

Let us now consider the projective 5-space P^5. The points of this space are ratios $(x_1 : x_2 : x_3 : x_4 : x_5 : x_6)$. The hypersurface of second degree defined in P^5 by the equation

$$x_1x_4 + x_2x_5 + x_3x_6 = 0$$

is clearly homeomorphic to the 4-manifold of all lines of projective 3-space.

REMARK. Every domain (connected open set) of a closed n-manifold is an example of an open n-manifold. However by no means every open n-manifold is homeomorphic to a domain of a closed manifold. The open 2-manifold pictured in Fig. 8 is not homeomorphic to any subset of any closed 2-manifold.

GENERAL REMARK CONCERNING MANIFOLDS. The following fundamental theorem will be proved in Chapter V:

A topological space cannot be, for $n \neq m$, both an n-manifold and an m-manifold.

§6. Partially ordered sets and discrete spaces [31]

§6.1. Definitions.

DEFINITION 6.11. A set Θ of arbitrary elements is said to be partially ordered if an order is defined in Θ, i.e., if a rule is given by which it is possible to determine, for any pair of elements, whether one of the pair follows (or precedes) the other. The order must satisfy the following conditions (order axioms):

1. If b follows a, then a does not follow b.
2. If b follows a, and c follows b, then c follows a.

DEFINITION 6.12. A partially ordered set Θ is said to be (*simply*) *ordered* if, for every pair of distinct elements a and b of Θ, either a follows b or b follows a.

REMARK 1. If b follows a, we also say that a precedes b and write $b > a$ or $a < b$.

REMARK 2. The first order axiom asserts that the relation $a < b$ excludes the relation $a > b$. Hence, in particular, $a < a$ cannot occur, since this would imply both $a < a$ and $a > a$.

DEFINITION 6.13. A (1–1) mapping f of a partially ordered set Θ_1 onto a partially ordered set Θ_2 is said to be a *similarity mapping* (or *transformation*) if f is order preserving, i.e., if

$$f(p) > f(p') \quad \text{in} \quad \Theta_2$$

is equivalent to $p > p'$ in Θ_1. Two partially ordered sets Θ_1 and Θ_2 are said to be *similar* if there is a similarity transformation which maps one onto the other.

§6.2. Examples of partially ordered sets.

1. *Any set of simplexes or, in general, convex polyhedral domains* (see Appendix 1, §3) is partially ordered by defining $T_2 > T_1$, where T_1 and T_2 are any two polyhedral domains, to mean that T_1 is a proper face of T_2. This is called the *geometric order* in the set of convex polyhedral domains.

2. The set consisting of the points, straight lines, and planes of a Euclidean space is partially ordered by letting a point (straight line) precede any straight line (plane) containing it.

3. The set Θ of real functions defined on a point set X is partially ordered if $f_1 > f_2$ is taken to mean that $f_1(x) \geq f_2(x)$ for all $x \in X$ and $f_1(x) > f_2(x)$ for at least one $x \in X$.

The following example is well known and fundamental:

4. *The partially ordered set of subsets of an arbitrary set M.* Let Θ be a set whose elements are subsets A_α of a set M,

$$\Theta = \{A_\alpha\}, \qquad A_\alpha \subseteq M.$$

If A_α and A_β are any two elements of Θ, put $A_\alpha > A_\beta$ if A_β is a proper subset of A_α, i.e., if $A_\alpha \supset A_\beta$. The resulting order is called the *natural order* in the collection of sets Θ; it obviously converts Θ into a partially ordered set.

5. If M consists of $n + 1$ elements, the set of all nonempty subsets of M, with the natural order, is similar to the set of *all faces of an n-simplex*, with the geometric order.

§6.3. The sets $A_\Theta p$ and $O_\Theta p$. Let Θ be a partially ordered set and let $p \in \Theta$.

Denote by $A_\Theta p$ the set of all $x \in \Theta$ such that $x \leq p$. The set $A_\Theta p$ is called the *combinatorial closure* of p in Θ. Clearly, $p \in A_\Theta p$.

The set of all $x \in \Theta$ such that $x \geq p$ will be denoted by $O_\Theta p$. This set is called the *star* of p in Θ. Obviously, $p \in O_\Theta p$.

EXAMPLES. 1. Let Θ be the partially ordered set of all points, straight lines, and planes of three-dimensional space. If p is a point of Θ, $A_\Theta p$ consists of just the point p; if p is a straight line, $A_\Theta p$ is the set consisting of the straight line p and all points lying on this line; if p is a plane, $A_\Theta p$ is the plane p and all lines and points lying on this plane. In an analogous fashion, $O_\Theta p$ consists of the point p and all straight lines and planes passing through p, if p is a point; of the straight line p and all planes containing p, if p is a straight line; and of just the plane p, if p is a plane.

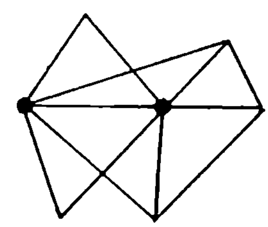


FIG. 9

2. Let Θ be the set of all triangles, sides, and vertices sketched in Fig. 9. If $p \in \Theta$, $O_\Theta p$ consists of all simplexes having p as a face. For example, for the case shown in Fig. 9:

If p is a triangle, $O_\Theta p$ consists of the single element p, and $A_\Theta p$ is the triangle p, its sides, and vertices; if p is a side, $O_\Theta p$ consists of this side and of the triangles having p as a side (in the figure there are one, two, or four such triangles, depending on the side chosen), and $A_\Theta p$ consists of the side p and its endpoints; if p is a vertex, $O_\Theta p$ consists of the triangles having p as a vertex and of the sides having p as an endpoint; while $A_\Theta p$ is just the vertex p.

THEOREM 6.3. *Every partially ordered set Θ is similar to its set $\mathfrak{A}$ of subsets $\{A_\Theta p\}$, $p \in \Theta$, partially ordered by the natural order.*

The proof is left to the reader as a simple exercise.

§6.4. Duality of partially ordered sets. Let Θ be a partially ordered set and let Θ' be the partially ordered set with the same elements as those of Θ but with the converse order relation, i.e., $a < b$ in Θ' if $a > b$ in Θ. The partially ordered set Θ' is called the *dual* of Θ. The relation of duality is clearly symmetric: if Θ' is dual to Θ, then Θ is dual to Θ'.

The closures of elements in Θ' are synonymous with the stars of elements in Θ and conversely: $A_{\Theta'} p = O_\Theta p$, $O_{\Theta'} p = A_\Theta p$.

We have seen that every partially ordered set is similar to the set of closures of its elements naturally ordered. Therefore, Θ', the dual of Θ, is similar to the set of stars of the elements of Θ with the natural order.

EXAMPLES. 1. Let Θ consist of the faces, edges, and vertices of a cube with the geometric partial order. Its dual Θ' is similar to the set of faces,

edges, and vertices of an octahedron also with the geometric partial order.

2. The partially ordered set of all proper faces of an n-simplex geometrically ordered is similar to its dual. To realize the correspondence, it suffices to associate with each face of the simplex the face opposite it.

3. Let us partition three-dimensional space into congruent cubes of side 1 and with vertices having integral coordinates. Let Θ consist of all these cubes, their faces, edges, and vertices geometrically ordered. The dual of Θ is similar to Θ. To realize a similarity transformation, it suffices to displace the whole system of cubes along the vector with components $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and to associate with each element p' of the displaced system the element p of the old system having a common center with p'.

§6.5. Discrete spaces.

Definition 6.51. A T_0-space is called a *discrete space* if the union of an *arbitrary* number of closed sets of the space is closed (or, equivalently, if the intersection of *any* number of open sets of the space is open).

Theorem 6.52. *We shall say that a subset A of a partially ordered set Θ is closed if $p \in A$ and $p' < p$ imply $p' \in A$.*

This topology converts Θ into a discrete space $R = f(\Theta)$. Conversely, every discrete space R can be turned into a partially ordered set $\Theta = \varphi(R)$ if, for any two distinct elements $p, p' \in R$,

$$p' < p$$

is taken to mean that

$$p' \in \bar{p}.$$

It follows that

$$f[\varphi(R)] = R$$

and

$$\varphi[f(\Theta)] = \Theta.$$

The proof consists of a routine verification of the axioms of a discrete space and of a partially ordered set and may be left to the reader.

In accordance with Theorem 6.52, partially ordered sets can be identified with discrete spaces. Then the combinatorial closure of an element p of Θ is synonymous with the closure in the space $f(\Theta)$ of the set consisting of the single point p, and the star of $p \in \Theta$ is the minimal neighborhood of the point p in $f(\Theta)$, i.e., the intersection of all open sets of the space $f(\Theta)$ containing p.

The transition from a partially ordered set Θ to its dual Θ' corresponds

to the transition from the discrete space R to its dual space R', whose closed sets are the open sets of the space R (and conversely).

A *similarity mapping* of one partially ordered set onto another is, therefore, equivalent to a *topological mapping* of the corresponding discrete spaces.

The identification of partially ordered sets with discrete spaces hence enables us to carry over to partially ordered sets the various theorems proved for topological spaces, e.g., *all theorems on connectedness.*

§7. Complete metric spaces and compacta [32]

§7.1. Definitions and simplest properties of complete metric spaces.

7.11. *A sequence of points*

$$x_1, x_2, \cdots, x_n, \cdots$$

of a metric space R is called a fundamental (or Cauchy) sequence if for every $\epsilon > 0$ there is a natural number $n(\epsilon)$ such that $p \geq n(\epsilon)$, $q \geq n(\epsilon)$ imply that $\rho(x_p, x_q) < \epsilon$.

It is obvious that every convergent sequence is a fundamental sequence.

7.12. *A metric space is said to be complete if every one of its fundamental sequences converges.*

Of the properties of complete metric spaces we note the following two:

7.13. *In a complete metric space R every decreasing sequence of closed sets*

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

whose diameters approach zero has an intersection consisting of one point.

Indeed, since the diameters of the sets A_n approach zero, $\bigcap A_n$ cannot contain two points. On the other hand, if $a_n \in A_n$, then $\{a_n\}$ is a fundamental sequence whose limit is contained in $\bigcap A_n$.

7.14. *If $\Gamma_1, \Gamma_2, \cdots, \Gamma_n, \cdots$ is a sequence of open sets of a complete metric space R and each Γ_i is dense in R, then $E = \bigcap \Gamma_n$ is also dense in R.*

The proof of Theorem 7.14 is based on the following

LEMMA. *If an open set Γ of a metric space R is dense in R, then every open set $\Gamma_0 \subseteq R$ contains an open set Γ'* (of arbitrarily small diameter) *whose closure is contained in $\Gamma_0 \cap \Gamma$, i.e., $\bar{\Gamma}' \subseteq \Gamma_0 \cap \Gamma$.*

In fact, since Γ is dense in R, Γ_0 contains a point p of Γ; and, since $A = (R \setminus \Gamma_0) \cup (R \setminus \Gamma) = R \setminus (\Gamma \cap \Gamma_0)$ is closed, $\rho(p, A) > 0$.

Let $\epsilon > 0$ be such that $\epsilon < \rho(p, A)$, and set $\Gamma' = S(p, \epsilon)$. Every point of $\bar{\Gamma}'$ is at a distance $\leq \epsilon$ from p and is, therefore, contained in $\Gamma \cap \Gamma_0$. This proves the lemma.

We shall now prove Theorem 7.14. To do this, it suffices to show that there is a point of E in every open set $\Gamma \subset R$.

Let Γ'_1 be an open set of diameter <1 such that $\bar{\Gamma}'_1 \subseteq \Gamma_1 \cap \Gamma$, and suppose that the sets

$$\Gamma'_1 \supseteq \cdots \supseteq \Gamma'_n, \qquad \delta(\Gamma'_i) < (\tfrac{1}{2})^i \qquad (1 \leq i \leq n),$$

satisfying the conditions

$$\bar{\Gamma}'_1 \subseteq \Gamma_1, \cdots, \bar{\Gamma}'_n \subseteq \Gamma_n$$

have already been defined. Then, applying the lemma, construct a set Γ'_{n+1} of diameter $<(\frac{1}{2})^{n+1}$ such that $\bar{\Gamma}'_{n+1} \subseteq \Gamma'_n \cap \Gamma_{n+1}$.

The intersection $\bigcap \bar{\Gamma}'_n$ is nonempty (it consists of one point) and is contained in $\Gamma \cap E$. This proves the theorem.

Well known examples of complete metric spaces are: the real line, Euclidean n-space, Hilbert space, etc.

§7.2. ϵ-nets in compacta. Let R be a metric space. A finite set $N \subseteq R$ is called an *ϵ-net* if it has the property that every point x of R is at a distance $< \epsilon$ from some point of N.

If, for a given ϵ, R has no ϵ-net, there exists in R a sequence of points whose mutual distance from each other is $\geq \epsilon$.

Indeed, let a_1 be a point of R. Then a_1 is not an ϵ-net, since R has no ϵ-net. Hence there exists a point a_2 whose distance from a_1 is $\geq \epsilon$. The set consisting of a_1, a_2 is again not an ϵ-net. Hence there exists a point a_3 whose distance from a_1 and a_2 is $\geq \epsilon$. Continuing in this way, we obtain a sequence

$$a_1, a_2, \cdots, a_n, \cdots \tag{7.21}$$

of points of R whose mutual distance from each other is $\geq \epsilon$.

REMARK 1. Since the sequence (7.21) clearly does not have a limit point, we have proved the proposition:

7.21. *A compactum has an ϵ-net for every $\epsilon > 0$.*

DEFINITION 7.22. *A metric space is said to be totally bounded if it contains an ϵ-net for every $\epsilon > 0$.*

REMARK 2. The sequence (7.21) obviously has the property that no subsequence of (7.21) is a fundamental sequence. Hence, if a metric space R is not totally bounded, it contains a sequence which has no fundamental subsequence.

On the other hand, if R is totally bounded and

$$N(\epsilon) = \{a_1, \cdots, a_s\}$$

is an $(\epsilon/2)$-net of R, then the sets $S(a_i, \epsilon/2)$ cover R. Since there are only a finite number of these sets and their diameters are $\leq \epsilon$, every infinite sequence

$$a_1, a_2, \cdots, a_n, \cdots \tag{7.22}$$

of points of R contains an infinite subsequence whose diameter is less than an arbitrary preassigned $\epsilon > 0$.

Let $\epsilon_n \to 0$ and let

$$a_{i_1}, \cdots, a_{i_k}, \cdots \tag{7.22$_1$}$$

be a subsequence of (7.22) of diameter $< \epsilon_1$. Now construct a subsequence (7.22$_2$) of (7.22$_1$) of diameter $< \epsilon_2$, etc. The diagonal sequence of all these sequences is a fundamental subsequence of (7.22).

Hence,

7.23. *A metric space is totally bounded if, and only if, every sequence of points in the space contains a fundamental subsequence.*

Since every compactum is clearly a complete metric space, 7.21 and 7.23 yield

7.24. *In order that a metric space be a compactum, it is necessary and sufficient that it be complete and totally bounded.*

7.25. *Every totally bounded metric space R contains a countable set dense in R, i.e., R has a countable basis.*

In fact, if N_k is an ϵ_k-net and $\epsilon_k \to 0$ as $k \to \infty$, then the countable set $N = \bigcup N_k$ is dense in R.

§7.3. The space of continuous mappings. Let X be a topological space and let Y be a bounded (i.e., of finite diameter) metric space. Denote by $\mathfrak{C}(X, Y)$ the metric space defined in the following way. The points of $\mathfrak{C}(X, Y)$ are the continuous mappings of X into Y. The distance between two points $C_1 \in \mathfrak{C}(X, Y)$ and $C_2 \in \mathfrak{C}(X, Y)$ is defined as the least upper bound of $\rho[C_1(x), C_2(x)]$, $x \in X$.

7.31. *If Y is a compactum and X is any topological space, the space $\mathfrak{C}(X, Y)$ is a complete metric space.*

Proof. Let

$$C_1, C_2, \cdots, C_n, \cdots \tag{7.31}$$

be a fundamental sequence of points of $\mathfrak{C}(X, Y)$. This means that for every $\epsilon > 0$ there is an $n(\epsilon)$ such that

$$\rho[C_p(x), C_q(x)] < \epsilon \tag{7.32}$$

for all $x \in X$ and arbitrary $p > n(\epsilon)$, $q > n(\epsilon)$. For every $x \in X$ the sequence of points

$$C_1(x), C_2(x), \cdots, C_n(x), \cdots \tag{7.33}$$

is, under these conditions, a fundamental sequence, and hence, in virtue of the compactness of Y, it is a convergent sequence in Y. It then follows from (7.32) that the sequence of mappings $\{C_n\}$ converges uniformly to a con-

tinuous function $C \in \mathfrak{C}(X, Y)$. The sequence (7.31) obviously converges in the metric space $\mathfrak{C}(X, Y)$, to $C \in \mathfrak{C}(X, Y)$. This proves the theorem.

§7.4. Deformations. Homotopy classes of mappings of a compactum X into a compactum Y. Let C_0 and C_1 be two continuous mappings of a compactum X into a compactum Y. Denote by Π the topological product of the compactum X and the segment $I \equiv a \leq \theta \leq b$ of the real line. The points of the space $X \times I = \Pi$ are the pairs (x, θ), $x \in X$, $\theta \in I$. A continuous mapping $C(x, \theta)$ of Π into Y is called a *deformation* of the mapping C_0 into the mapping C_1 if

$$C(x, a) = C_0(x), \qquad C(x, b) = C_1(x)$$

for all $x \in X$. Two continuous mappings C_0 and C_1 of the compactum X into the compactum Y are said to be *homotopic* if there exists a deformation $C(x, \theta)$ of one into the other.

Since the homotopy relation is reflexive, symmetric, and transitive, the set of all continuous mappings of a compactum X into a compactum Y is partitioned into classes of homotopic mappings or, briefly, into *homotopy classes*.

REMARK. It is customary to put $a = 0$ $b = 1$, to write $C_\theta(x)$ instead of $C(x, \theta)$, and to speak of a *family of continuous mappings* $C_\theta(x)$ of a compactum X into a compactum Y, *depending continuously on the parameter* θ, $0 \leq \theta \leq 1$, or briefly, of a continuous family $C_\theta(x)$ of continuous mappings of X into Y.

§8. Coverings of normal spaces and, in particular, of compacta

§8.1. Closed and open coverings of topological spaces. The method of combinatorial topology. We already know that a (finite) system of subsets of a set M whose union is M is called a (finite) covering of M. In the sequel, we shall consider coverings of a topological space, namely, *open coverings*, consisting of open sets, and *closed coverings*, consisting of closed sets. *In all cases, unless otherwise noted, we shall consider only finite coverings.* The *order of a covering* α is the greatest integer n for which there exist n elements of the covering α having a nonempty intersection.

DEFINITION 8.11. *A system of sets β is called a refinement of a system of sets α if every element of β is contained in at least one element of α.*

A covering β of a set M is said to *follow* a covering α of M if β is a refinement of α and α is not at the same time a refinement of β. This relation will be written as "$\beta > \alpha$". It is easily seen that the relation "$>$" between the coverings of a set M is a partial order and thus converts the set of coverings of M into a partially ordered set. Whenever we speak of a partially ordered set of coverings of a set M, we will always have in mind the order defined above.

Combinatorial topology, in the broad sense, studies the properties of topological spaces by investigating the properties of the partially ordered sets of their open (or closed) coverings.

§8.2. Similar coverings.

Definition 8.21. Two finite indexed systems of sets

$$\alpha = \{A_1, A_2, \cdots, A_s\},$$
$$\beta = \{B_1, B_2, \cdots, B_s\}$$

are said to be *similar* if $\bigcap_{j=1}^{k} A_{i_j} \neq 0$ is equivalent to $\bigcap_{j=1}^{k} B_{i_j} \neq 0$, where the subscript i_j stands for any of the integers $1, \cdots, s$.

The equivalence existing in general combinatorial topology between the use of partially ordered sets of open and closed coverings is based on the following two theorems:

8.22. *If*

$$\alpha = \{A_1, \cdots, A_s\}$$

is a system of closed sets of a normal space R, there exists a system of open sets

$$\omega = \{O_1, \cdots, O_s\}$$

such that $A_i \subseteq O_i$ $(1 \leq i \leq s)$, and such that α is similar to both ω and $\bar{\omega} = \{\bar{O}_1, \cdots, \bar{O}_s\}$.

A very simple proof of this theorem is given in 8.3 for the case of a compactum R.

8.23. *For every open covering*

$$\omega = \{O_1, \cdots, O_s\}$$

of a normal space R there is a closed covering

$$\alpha = \{A_1, \cdots, A_s\}$$

of the space such that

$$A_i \subseteq O_i \qquad (1 \leq i \leq s).$$

Remark. As will be evident from the proof, we may even suppose in Theorem 8.23 that the A_i are the closures of open sets forming a covering of R.

Proof of 8.22. Let A be the union of all possible sets of the form $\bigcap_{j=1}^{k} A_{i_j}$, where the sets A_{i_j} $(1 \leq j \leq k)$ are such that $A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} = 0$.

Since A is closed and $A \cap A_1 = 0$, $R \setminus A$ is a neighborhood of A_1. Consequently, by 4.141, there exists an open set O_1 such that $A_1 \subseteq O_1$, $\bar{O}_1 \subseteq R \setminus A$.

The system of sets

$$\alpha = \{A_1, \cdots, A_s\}$$

is similar to the system

$$\alpha_1 = \{\bar{O}_1, A_2, \cdots, A_s\}.$$

Indeed, if $A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} \neq 0$, then, a fortiori, $\bar{O}_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} \neq 0$. Conversely, $A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} = 0$ implies that $A_{i_1} \cap \cdots \cap A_{i_k} \subseteq A$ which in turn implies that

$$\bar{O}_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} \subseteq \bar{O}_1 \cap A = 0.$$

Let us suppose that the open sets $O_i \supseteq A_i$ $(1 \leq i \leq r < s)$ have been constructed in such a way that α is similar to

$$\alpha_r = \{\bar{O}_1, \cdots, \bar{O}_r, A_{r+1}, \cdots, A_s\}.$$

Applying the same reasoning to the system α_r and the set A_{r+1} which we have just applied to the system α and the set A_1, we obtain an open set $O_{r+1} \supseteq A_{r+1}$ such that the systems α_r and

$$\alpha_{r+1} = \{\bar{O}_1, \cdots, \bar{O}_{r+1}, A_{r+2}, \cdots, A_s\},$$

and, by the same token, the systems α and α_{r+1}, are similar. Continuing in this way, we arrive at a system $\alpha_s = \{\bar{O}_1, \cdots, \bar{O}_s\}$ which satisfies the prescribed requirements.

Proof of 8.23. Corresponding to the system

$$\alpha' = \{R \setminus O_1, \cdots, R \setminus O_s\},$$

there exists, according to the proof above, a system of open sets

$$\omega' = \{O'_1, \cdots, O'_s\}$$

such that $R \setminus O_i \subseteq O'_i$ and such that the system $\{\bar{O}'_1, \cdots, \bar{O}'_s\}$ is similar to the system α'. Because of this similarity, it follows, in particular, that

$$\bar{O}'_1 \cap \cdots \cap \bar{O}'_s = 0.$$

Putting $O''_i = R \setminus \bar{O}'_i$, we obtain (since $R \setminus O'_i$ is closed)

$$\bar{O}''_i = \overline{R \setminus \bar{O}'_i} \subseteq \overline{R \setminus O'_i} = R \setminus O'_i.$$

Since $R \setminus O_i \subseteq O'_i$, this implies that

$$\bar{O}''_i \subseteq R \setminus (R \setminus O_i) = O_i.$$

Finally, since

$$O''_1 \cup \cdots \cup O''_s = R \setminus (\bar{O}'_1 \cap \cdots \cap \bar{O}'_s) = R,$$

the required covering is

$$\alpha = \{A_1, \cdots, A_s\},$$

where $A_i = \bar{O}''_i$.

§8.3. ϵ-coverings of compacta. Lebesgue numbers of a covering.

DEFINITION 8.31. *A covering of a metric space is called an ϵ-covering if the elements of this covering are of diameter $<\epsilon$.*

LEBESGUE'S LEMMA 8.32. *Let*

$$\alpha = \{A_1, \cdots, A_s\}$$

be a closed covering of a compactum Φ. Then there is a positive number δ with the following property: if $M \subset \Phi$ has diameter $\leq \delta$ and

$$A_{i_j} \in \alpha \qquad (1 \leq j \leq k) \tag{8.321}$$

are such that $M \cap A_{i_j} \neq 0$ $(1 \leq j \leq k)$, then $\bigcap_{j=1}^{k} A_{i_j} \neq 0$.

Proof. The proof is by contradiction. If there is no δ with the required property, then for every natural number n there is a finite sequence of elements (8.321) of α and a set $M_n \subset \Phi$ of diameter $\leq 1/n$ such that

$$M_n \cap A_{i_j} \neq 0 \quad (1 \leq j \leq k) \quad \text{and} \quad \bigcap_{j=1}^{k} A_{i_j} = 0.$$

Since the number of all combinations (8.321) of different elements of the covering α is finite, there is at least one combination (8.321) which, in the above sense, corresponds to an infinite number of different n's. Hence there is a sequence of natural numbers $\{n_j\}$ and a sequence of sets $\{M_{n_j}\}$, $j = 1, 2, \cdots$, of diameter $\leq 1/n_j$ which intersect all of the sets (8.321) and such that $\bigcap_{j=1}^{k} A_{i_j} = 0$. Let $p_{n_j} \in M_{n_j}$. Passing, if necessary, to a subsequence of $\{p_{n_j}\}$, we may suppose that the sequence $\{p_{n_j}\}$ converges to a point $p \in \Phi$. There are clearly points of each of the sets A_{i_j} $(1 \leq j \leq k)$ in every neighborhood of p, and since these sets are closed,

$$p \in A_{i_j} \qquad (1 \leq j \leq k).$$

Therefore, $\bigcap_{j=1}^{k} A_{i_j} \neq 0$, a contradiction. This proves the lemma.

DEFINITION 8.33. *Every positive number δ which satisfies the conditions of Lebesgue's lemma for a given closed ϵ-covering α of a compactum Φ is called a Lebesgue number of the covering.* Hence any sufficiently small positive number is a Lebesgue number of a closed covering. We shall, in addition, impose one more condition on the Lebesgue numbers of an ϵ-covering: we shall require that they be less than one half the difference between ϵ and the maximum of the diameters of the sets A_i. This definition of a Lebesgue number immediately implies

8.331. *If 2σ is a Lebesgue number of the closed ϵ-covering*

$$\alpha = \{A_1, \cdots, A_s\}$$

of the compactum Φ and if $O_i = S(A_i, \sigma)$ is a σ-neighborhood of the set A_i, then $\omega = \{O_1, \cdots, O_s\}$ and $\bar{\omega} = \{\bar{O}_1, \cdots, \bar{O}_s\}$ are, respectively, open and closed ϵ-coverings of Φ which are both similar to the covering α.

LEMMA 8.34. *If*

$$\omega = \{O_1, \cdots, O_s\}$$

is an open covering of a compactum Φ, *there is a positive number* $\eta = \eta(\omega)$, *with the following property: every set* $M \subset \Phi$ *of diameter* $< \eta$ *is wholly contained in at least one of the sets of the covering* ω.

Proof. Let η be the greatest lower bound of the diameters of all sets $M \subset \Phi$ which are not contained in any element of the covering ω. We shall show that η is positive and this will prove the lemma. Let

$$M_1, M_2, \cdots, M_n, \cdots$$

be any sequence of such sets whose diameters approach η. Let $p_n \in M_n$. Passing, if necessary, to a subsequence of $\{p_n\}$, we may suppose that $\{p_n\}$ converges to a point $p \in \Phi$. The point p is contained in an element O_i of the covering ω and consequently has a positive distance 2ϵ from the closed set $R \setminus O_i$, so that

$$S(p, 2\epsilon) \subseteq O_i. \tag{8.341}$$

By definition, no M_n is contained in O_i and, a fortiori, in $S(p, 2\epsilon)$. Since $p_n \in S(p, \epsilon)$ for all sufficiently large n, it follows that for all such n the diameter of M_n must necessarily be $> \epsilon$ [since, otherwise, $M_n \subseteq S(p, 2\epsilon)$]. However, since η is the greatest lower bound of the diameters of the sets M_n, $\eta \geq \epsilon$. This proves the lemma.

We shall conclude this preliminary investigation of coverings with the following very general and quite elementary proposition which will be convenient in the sequel.

8.35. *Let* $\mathfrak{E}$ *be a property of closed coverings* (for example, the property of having a given order). *In order that there exist, for every open covering* ω *of a compactum* Φ, *a closed covering which is a refinement of* ω *and which has property* $\mathfrak{E}$, *it is necessary and sufficient that there exist, for every* $\epsilon > 0$, *a closed* ϵ*-covering of the compactum* Φ *with property* $\mathfrak{E}$.

Proof of necessity. For any $\epsilon > 0$, take any open ϵ-covering and then a closed covering which is a refinement of the open covering, and which has property $\mathfrak{E}$. This yields a closed ϵ-covering with property $\mathfrak{E}$.

Proof of sufficiency. If, for every $\epsilon > 0$, there exist ϵ-coverings with property $\mathfrak{E}$, to obtain a closed covering with property $\mathfrak{E}$ which refines a given open covering ω, it suffices to determine the number $\eta = \eta(\omega)$ of Lemma 8.34 for the covering ω and to take an arbitrary closed η-covering α with property $\mathfrak{E}$. According to the definition of η, the covering α is a refinement of ω.

§8.4. Definition of dimension.

DEFINITION 8.41. *The dimension of a bicompactum* Φ, *denoted by* $\dim \Phi$, *is the least integer* n *such that every open covering of* Φ *has a refinement which is a closed covering of* Φ *of order* $\leq n + 1$.

If there is no least integer n for which this is true, $\dim \Phi = \infty$.

REMARK 1. If $\dim \Phi = \infty$ there exists for every n an open covering ω of Φ such that *every* closed covering which refines ω has order $>n + 1$.

EXERCISE. Prove that the definition of dimension may also be formulated as:

8.41′. The dimension of a bicompactum Φ is the least integer n for which every open covering of Φ has a refinement which is a *simple* closed covering of order $n + 1$.

REMARK 2. The dimension of the empty set shall, by definition, be equal to -1.

By 8.35, the definition of dimension for compacta can be phrased as:

8.42. *The dimension of a compactum Φ is the least integer n such that for every $\epsilon > 0$ there exists a closed ϵ-covering of Φ of order $\leq n + 1$.*

REMARK 3. The intuitive meaning of this definition is very simple. For $n = 2$, it asserts that every two-dimensional "area" (compactum) can be "paved" with arbitrarily small "bricks" (closed sets) in such a way that no point of the area is contained in more than three of these bricks, but that if the bricks are sufficiently small, at least three have a point in common. Similarly a "volume" can be filled in with sufficiently small bricks so that no point of the volume is in more than four of the bricks, but if the bricks are sufficiently small at least four will have a point in common. Hence the theorem, known as the Pflastersatz (see V, Theorem I′ and 2.24), that, e.g., a square has dimension 2.

The perception of the fact that the order of a covering contains the concept of dimension was a great achievement of mathematical thought. This achievement is due to Lebesgue, who was the first to formulate and prove (not, however, completely without error) the Pflastersatz in 1911. The first complete proof of this theorem was given by Brouwer in 1913.

If there is no n satisfying 8.42, $\dim \Phi = \infty$, by definition. Then for every n there is an $\epsilon > 0$ such that every closed ϵ-covering is of order $>n + 1$.

Since open sets, closed sets, and the order of a covering are invariant under a topological mapping of a bicompactum Φ onto a bicompactum Φ', it follows that two homeomorphic bicompacta have the same dimension, i.e., *dimension is a topological invariant.*

A significant part of Chapter V and all of Chapter VI are devoted to the dimension theory of compacta. We shall close this chapter by deriving one property of dimension which is an immediate consequence of the definition; we shall prove it only for compacta, leaving the proof for bicompacta to the reader.

8.43. *If a compactum Φ_0 is contained in a compactum Φ*, $\dim \Phi_0 \leq \dim \Phi$.

Indeed, let $\epsilon > 0$ and let $\alpha = \{A_1, \cdots, A_s\}$ be a closed ϵ-covering of the compactum Φ. Then

$$\alpha_0 = \{\Phi_0 \cap A_1, \cdots, \Phi_0 \cap A_s\}$$

is an ϵ-covering of the compactum Φ_0 whose order, obviously, does not exceed the order of α. This proves 8.43.

REMARK 4. Theorems 8.22 and 8.23 imply

8.44. *The dimension of a bicompactum Φ may be defined as the least integer n satisfying the condition:*

Every open covering of the bicompactum Φ can be refined by an open covering of Φ of order $\leq n + 1$.

In particular, the dimension of a compactum Φ can be defined as the least integer n such that for every $\epsilon > 0$ there exists an open ϵ-covering of Φ of order $\leq n + 1$.

REFERENCES*

1. General references: [A–H; I, II]; [H; VII–IX]: [H*; VI, VIII]; [L; I]; [N; II–IV]; [P; II]; [Si]; [W; I–II, VII (§§1–3)]; also the relevant sections of [Wi; I–III].
2. [A–H; I (§§1–3, 7, 8)]; [L; I (§§1, 2, 8)]; [P; II (§§7–10, 12, 15)]; [Si; II, VI].
3. [Si, p. 122].
4. [Si; p. 128, Theorem 74].
5. [N; p. 71, Theorem 1.1 and Corollary].
6. [A–H; I (§§4–6), II (§1)]; [L; I (§§5–6)].
7. See [Si; p. 105, Theorem 63] for a proof of a stronger theorem.
8. See [4] and [7].
9. ALEKSANDROV [g; §5, Theorem 2].
10. ALEKSANDROV [g, §3].
11. ALEKSANDROV [g, §3]; [A–H; II (§§2, 3)]; [L; I (§§5, 6, 8)]; [Si; V].
12. For necessity see [Si; p. 95, Urysohn's Lemma]; for sufficiency see [L; p. 27, 34.1].
13. [L; p. 28, 34.2]; [A–H; p. 73, Satz VIII; proof on p. 75]. For the special case of a metric space see [A–H; pp. 76–78].
14. Bicompactness in the original. The term "compact" has been substituted for "bicompact" since the book deals mostly with metric spaces and the two notions are equivalent in such spaces (for a proof see [N; p. 47, Theorem 15.3, and p. 204, Note 10]; also [L; I, (§5, 21)]).
15. [Si; p. 117, Theorem 70].
16. [N; p. 42, Theorem 14.3].
17. [Wi; p. 35, Lemma 12.10].
18. [Wi; p. 69, Lemma 1.1].
19. [Wi; p. 74, Theorem 1.27].
20. [L; p. 39, Theorem 46.4].
21. [L; p. 19, Theorem 24.1, and p. 24, Theorem 30.2].
22. See [21] and [L; p. 35, 43.8–43.10].
23. [Wi; p. 74, Theorem 2.1].
24. ALEKSANDROV [g, §5].
25. [Wi; p. 69, Lemma 1.2].
26. Apply 4.51.
27. [Wi; p. 74, Theorem 1.28].
28. [Si; p. 153, Theorem 80].
29. ALEKSANDROV [a]; [A–H; I (§5), II (§2)]; [W, pp. 122–127].
30. [B; VII (§35)].
31. ALEKSANDROV [c; g, §6].
32. [A–H; II (§4)]; [H*, §26]; [L; pp. 37–38, 45]; [N; pp. 17, 18, 51]; [Si, VII].

* For abbreviations see the Bibliography.

Chapter II

THE JORDAN THEOREM

The proof of the Jordan theorem given in this chapter is due to E. Schmidt (see Schmidt [a]). This proof is quite elementary and requires no preliminary knowledge of topology except for the simplest facts about connectedness (I, 3). These are needed to understand the theorem itself. A second merit of Schmidt's proof is that the auxiliary constructions on which the proof rests (the angle function, the index of a point with respect to a continuous mapping, etc.) have a wide range of application far beyond the bounds of the Jordan theorem itself. They are to be numbered among those elementary concepts with which every student of, e.g., the theory of differential equations or the theory of functions of a complex variable needs to be acquainted.

§1. Formulation of the Jordan theorem. Common boundaries of domains

§1.1. Formulation of the Jordan theorem. A compactum Φ homeomorphic to a circumference is called a *simple closed curve* or (*closed*) *Jordan curve*. A compactum homeomorphic to a straight line segment is called a *simple* or *Jordan arc*. This chapter is devoted to a proof of the following important and celebrated theorem:

THE JORDAN THEOREM. *Let Φ be a Jordan curve in the plane R^2. The complementary open set $R^2 \setminus \Phi$ consists of two disjoint domains (connected open sets) Γ_0 and Γ_1 whose common boundary is Φ.*

In other words:

1. $R^2 \setminus \Phi = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = 0$;
2. Γ_0 and Γ_1 are domains;
3. $\Phi = \bar{\Gamma}_0 \setminus \Gamma_0 = \bar{\Gamma}_1 \setminus \Gamma_1$.

REMARK 1. A closed set A of a connected topological space R is said to *separate* (or *disconnect*) R if the open set $R \setminus A$ is not connected; but if $R \setminus A$ is connected, then A, by definition, does *not separate* R.

REMARK 1_0. A space is separated by the empty set if, and only if, it is not connected.

It is customary to state the Jordan theorem in the following more compact way:

Every plane Jordan curve Φ separates the plane R^2 into two domains whose common boundary is Φ.

The Jordan theorem contains three assertions:

1. The open set $R^2 \setminus \Phi$ consists of two components, Γ_0 and Γ_1.
2. Each of these components is an open set.
3. Φ is the common boundary of the domains Γ_0 and Γ_1.

REMARK 2. With regard to assertion 2, it should be noted that the components of an open subset of a space R need not be open sets in R; for instance, if R is the Cantor perfect set (with its usual topology), each point $x \in R$ is its own component in R, i.e., the components of R are degenerate sets and none of them are open in R. This remark indicates the significance of Theorems 1.25 and 1.27.

We shall see in the sequel that the following theorem, also very important, is closely related to the Jordan theorem:

A plane Jordan arc Φ does not disconnect the plane R^2.

In other words:

If Φ is a Jordan arc in R^2, the open set $R^2 \setminus \Phi$ is connected.

These two theorems, both proved in this chapter, are special cases of a theorem which will be proved in Chapter XIV and again in Chapter XV.

§1.2. Domains in R^n and their boundaries.

Let us note first that:

1.21. Every convex open set in R^n, e.g., an open solid n-sphere, is connected by virtue of I, 3.12. Hence every set homeomorphic to a convex domain of R^n is also connected.

1.22. An open set $\Gamma \subseteq R^n$ is connected, if, and only if, every pair of points of Γ can be joined by a simple broken line (i.e., a broken line without multiple points) contained in Γ.

Proof. If every pair of points of Γ can be joined by a broken line (even one containing multiple points) in Γ, then Γ is connected by I, 3.12.

To prove the necessity of the condition, we shall show that if there are two points, a and b, in Γ which cannot be joined by a simple broken line in Γ, then Γ is not connected.

Let us denote by Γ_a the set of all points of Γ which can be joined to a by broken lines in Γ, and by Γ_b the set of all remaining points of Γ.

If p is any point of Γ, all points of R^n whose distance from p is less than $\epsilon = \rho(p, R^n \setminus \Gamma)$ are contained in Γ and can therefore be joined to p by a segment contained in Γ. It follows readily that Γ_a and Γ_b are open in Γ. Since $\Gamma_a \cap \Gamma_b = 0$ and $a \in \Gamma_a$, $b \in \Gamma_b$, Γ is not connected. This completes the proof of Theorem 1.22.

1.23. Let Γ be a domain of R^n, $n \geq 2$, and let $E \subset \Gamma$ be a finite set. Then the set $\Gamma \setminus E$ is connected, i.e., $\Gamma \setminus E$ is also a domain.

Indeed, let $a, b \in \Gamma \setminus E$ be joined by a broken line in Γ. An arbitrarily small displacement of the vertices of this broken line will bring it into general position (see Appendix 1) with respect to the finite set E. It will then contain no points of E. Hence the displaced broken line will join a and b in $\Gamma \setminus E$.

1.24. Let $\bar{E}^n$ be a closed solid n-sphere in R^n. The open set $R^n \setminus \bar{E}^n$ is connected.

In fact, an inversion with respect to the bounding sphere S^{n-1} of $\bar{E}^n$ maps $R^n \setminus \bar{E}^n$ into $E^n \setminus o$, where E^n is the corresponding open solid n-sphere and o its center. Since it has been shown that $E^n \setminus o$ is connected, $R^n \setminus \bar{E}^n$ is also connected.

1.25. The components of an open subset of R^n are open, i.e., are domains.

Before proving Theorem 1.25, let us make the following definition:

DEFINITION 1.26. A metric space R is said to be *locally connected at a point* p if p has connected neighborhoods of arbitrarily small diameter; R is *locally connected* if it is locally connected at all its points.

Since spherical neighborhoods in R^n are connected, it follows that R^n and all open sets in R^n are locally connected (Def. 1.26 implies that every open set of a locally connected space is also locally connected). The curve consisting of the segment $-1 \leq y \leq 1$ of the y-axis and of the graph of $y = \sin 1/x$, $0 < x \leq 1/\pi$ (Fig. 2, p. 15) is an example of a compactum which is not locally connected. This curve is not locally connected at any of its points on the axis of ordinates.

These remarks make it clear that 1.25 is a special case of the following important theorem:

1.27. *The components of a locally connected space are open sets* (hence domains) *of the space.*

Proof of Theorem 1.27. Let R be a locally connected space, Q a component of R, and p a point of Q. Denote by U any connected neighborhood of p. By I, 3.13, the set $Q \cup U$ is connected. Since Q is a component, $Q = Q \cup U$. Hence p is an interior point of Q. This proves the theorem.

1.28 If Φ is a compactum in R^n, $R^n \setminus \Phi$ has exactly one unbounded component.

Let $\bar{E}^n$ be a solid sphere containing Φ in its interior (such a sphere exists, since Φ is a bounded set). The set $R^n \setminus \bar{E}^n$ is connected. Hence it is contained in one component of $R^n \setminus \Phi$, which is, therefore, unbounded. All the remaining components of $R^n \setminus \Phi$ are contained in $\bar{E}^n$ and, consequently, are bounded.

DEFINITION 1.29. A compactum Φ is said to be an *absolute boundary* in R^n if Φ separates R^n and is the common boundary of all the components of the open set $R^n \setminus \Phi$.

DEFINITION 1.291. An absolute boundary in R^n is called a *regular boundary* if it separates R^n into precisely two domains.

Such curves as a circumference, an ellipse, etc., are regular boundaries of domains in R^2; the sphere and torus are examples of regular boundaries of domains in R^3. *The Jordan theorem asserts that every plane closed Jordan curve is a regular boundary in* R^2.

At first glance it would seem that every absolute boundary is regular: it is difficult to imagine, for instance, a plane closed set which separates

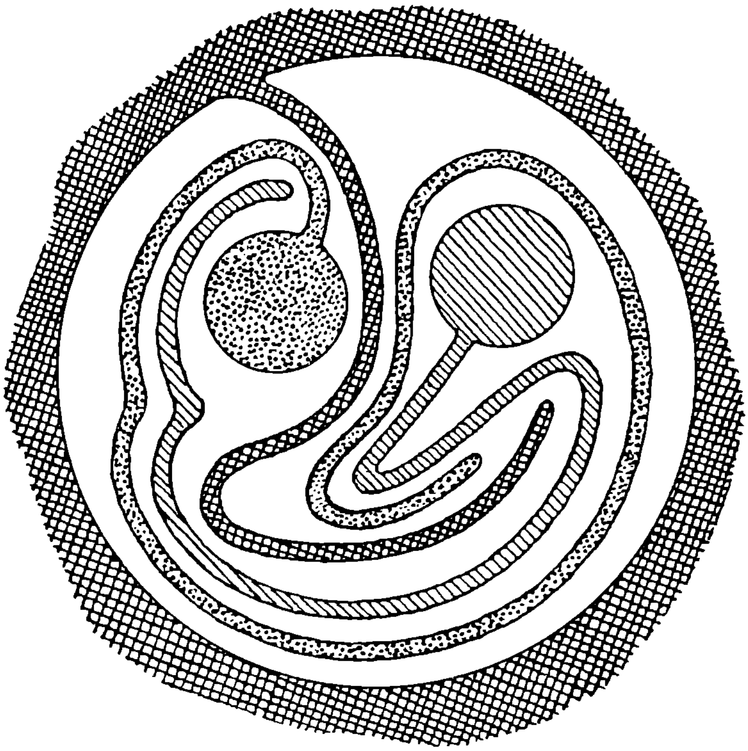


FIG. 10

the plane into more than two domains of which it is the common boundary. Nevertheless, such closed sets exist; they were first constructed by Brouwer in 1909. The following example (first published by the Japanese mathematician Yoneyama) is a slight modification of the original example of Brouwer.

Let us imagine an island Φ surrounded by a sea Γ. We shall represent the island by a circle (together with its circumference), so that the sea $\Gamma = R^2 \setminus \Phi$ is an open set. Let the island Φ have two circular lakes, one containing cold water and the other warm (each of the lakes is in the interior of the circle).

Let us imagine the following program. In the first hour, three dead-end canals are dug in the interior of the island, one leading out of the sea and the other two leading out of each lake. Each of these canals is made to wind in such a way that the distance of every point of dry land of the island is less than 1 mile both from the sea water and from the water of each of the lakes (Fig. 10).

In the next half hour, each one of the canals is extended in such a way that, at the end of this half hour, the distance of every point of dry land (remaining on the island) from the sea water and the water of both lakes is less than $\frac{1}{2}$ mile. In the following $\frac{1}{4}$ hour the canals are continued so

that the distance of every point of dry land from the sea, warm fresh water, and cold fresh water is made less than $\frac{1}{4}$ mile, etc. After $1 + \frac{1}{2} + \frac{1}{4} + \cdots + (\frac{1}{2})^n$ hours the work on all three canals is continued in a manner to make the distance of each point of dry land from the sea and each lake less than $(\frac{1}{2})^{n+1}$ miles (this will be accomplished in the next $(\frac{1}{2})^{n+1}$ hours). Throughout, the canals never touch each other or cross themselves, i.e., they remain dead-end canals. At the end of two hours, there remains of the island only a closed set Φ_0 which is nowhere dense in the plane. An arbitrary neighborhood of a point of Φ_0 contains points of the sea as well as points of the two lakes. Hence the closed set Φ_0 is the common boundary of three plane domains, the sea and both lakes.

An absolute boundary in the plane R^2 which separates R^2 into an arbitrary finite, or even denumerable, number of domains can be constructed by the same method.

REMARK. A compactum $\Phi \subset R^2$ which is the common boundary of two plane domains (two components of the open set $R^2 \setminus \Phi$) may not be an absolute boundary: the set Φ (Fig. 11) consisting of a circumference and a spiral curve winding inside it separates the plane into three domains, but is the boundary of only two of these domains (in Fig. 11 one of these domains has been left blank, the other is stippled; the third is the domain exterior to the circle and is cross-hatched).

THEOREM 1.2. *In order that a compactum Φ be an absolute boundary in*

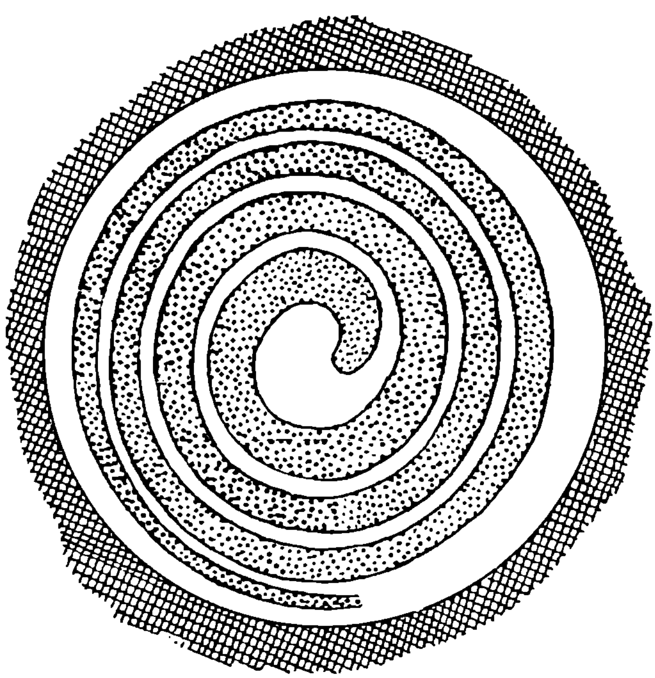


FIG. 11

R^n, it is necessary and sufficient that it separate the space R^n into at least two domains and that no proper closed subset Φ_0 of Φ have this property, i.e., if $\Phi_0 \subset \Phi$ is closed, then $R^n \setminus \Phi_0$ is connected.

Proof. Let Φ separate R^n into a countable number of domains (in a space with a countable basis, every collection of mutually disjoint open sets is countable at most) $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots$ and assume that Φ is the common boundary of all these domains.

Let $\Phi_0 \subset \Phi$. Since $\Phi = \bar{\Gamma}_i \setminus \Gamma_i$, it follows from I, 3.19 that each of the sets

$$\Gamma_i \cup (\Phi \setminus \Phi_0)$$

is connected. Hence, by I, 3.16 the set

$$\bigcup_i[\Gamma_i \cup (\Phi \setminus \Phi_0)] = R^n \setminus \Phi_0$$

is also connected. This proves the necessity of the condition.

Now let Φ separate R^n into domains $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots$ and suppose that no closed set $\Phi_0 \subset \Phi$ separates R^n. We shall show that then

$$\Phi = \bar{\Gamma}_k \setminus \Gamma_k$$

for every k. It is clear, to begin with, that no point $p \in \bar{\Gamma}_k \setminus \Gamma_k$ can be contained in any Γ_i. Hence $p \in \Phi$, so that

$$\Phi_k = \bar{\Gamma}_k \setminus \Gamma_k \subseteq \Phi.$$

Since $\Gamma_i \subset R^n \setminus \bar{\Gamma}_k$, $i \neq k$, it follows that $R^n \setminus \bar{\Gamma}_k$ is nonempty and that, therefore,

$$R^n \setminus \Phi_k = \Gamma_k \cup (R^n \setminus \bar{\Gamma}_k)$$

is the union of two nonvacuous disjoint open sets Γ_k and $R^n \setminus \bar{\Gamma}_k$. Hence $R^n \setminus \Phi_k$ is not connected, i.e., Φ_k separates R^n. Since $\Phi_k \subseteq \Phi$, our hypothesis implies that $\Phi_k = \Phi$ for all k. This completes the proof of the theorem.

§1.3. Outline of the proof of the Jordan theorem. Let us first make the following simple remark:

A plane Jordan arc or Jordan curve Φ is nowhere dense in the plane.

Indeed, every connected subset of Φ is separated by some pair of points of this subset; the interior of a circle is a connected set which is not separated by any of its finite subsets. Therefore, Φ does not contain the interior of any circle. Hence, since Φ is closed, it is nowhere dense in the plane.

In §2 we introduce a concept basic to the whole proof, the index of a point relative to a closed curve. In §3 we prove that a simple arc does not disconnect the plane, and §4 contains a proof of the fact that a closed Jordan curve separates the plane into two domains.

These two propositions imply the last assertion of the Jordan theorem: a Jordan curve Φ is the common boundary of the two components of the open set $R^2 \setminus \Phi$. To prove this it suffices, by 1.2, to show that no closed set $\Phi_0 \subset \Phi$ separates R^2. Let $\Phi_0 \subset \Phi$ and let ξ be any point on the curve Φ not contained in Φ_0. Since Φ_0 is closed, there exists an open arc (ab) of Φ containing the point ξ and free of points of Φ_0. The set $\Phi_1 = \Phi \setminus (ab)$ is a simple arc lying on Φ and containing Φ_0.

It is required to prove that $R^2 \setminus \Phi_0$ is connected. However, since every point of $\Phi_1 \setminus \Phi_0$ is a limit point of $R^2 \setminus \Phi_1$, the connectedness of $R^2 \setminus \Phi_0$ follows from that of $R^2 \setminus \Phi_1$ (I, 3.19), i.e.,

$$R^2 \setminus \Phi_1 \subseteq R^2 \setminus \Phi_0 = R^2 \setminus \Phi_1 \cup \Phi_1 \setminus \Phi_0,$$

which in turn is contained in the closure of $R^2 \setminus \Phi_1$.

Hence to complete the proof of the Jordan theorem, it is enough to prove that a Jordan curve separates the plane into two domains and that a simple arc does not disconnect the plane.

§1.4. Notation. Orientation of simple arcs and simple closed curves. We shall denote a simple arc with endpoints a and b by $[ab]$ and the corresponding open arc, i.e., the set $[ab] \setminus a \cup b$, by (ab). If $\bar{\Lambda}$ denotes a simple arc, then Λ will stand for the corresponding open arc.

Since a simple arc is a topological image of a segment, it is possible to speak of an oriented simple arc, i.e., an arc directed from a to b or from b to a. An oriented simple arc will be denoted by $[ab]^{\rightarrow}$ or $[ba]^{\rightarrow}$, respectively; an oriented open arc by $(ab)^{\rightarrow}$ or $(ba)^{\rightarrow}$, respectively.

The direction (or sense) from a to b on a simple arc $[ab]$ may be defined as follows. Let C be any topological mapping of the segment $0 \leq x \leq 1$ onto the simple arc $[ab]$ such that $C(0) = a$ and $C(1) = b$. The mapping C orders the points of the arc $[ab]$: if $y = C(x)$, $y' = C(x')$ are two points of $[ab]$, put $y < y'$ if $x < x'$ on $[01]$.

It remains to be shown that this order is independent of the choice of the mapping C.

Let C_1 and C_2 be topological mappings of the segment $[01]$ onto the arc $[ab]$ such that $C_1(0) = C_2(0) = a$, $C_1(1) = C_2(1) = b$. If C_1 and C_2 were to define different orders on $[ab]$, then, for some two points

$$y = C_1(x_1) = C_2(x_2) \in [ab]$$

$$y' = C_1(x'_1) = C_2(x'_2) \in [ab]$$

it would follow that, e.g.,

$$x_1 < x'_1, \qquad x_2 > x'_2. \tag{1.41}$$

Then $C = C_2^{-1}C_1$ is a topological mapping of the segment $[01]$ onto itself and

$$(1.42) \qquad C(0) = 0, \qquad C(1) = 1,$$
$$C(x_1) = x_2, \qquad C(x'_1) = x'_2.$$

However, C, as a topological mapping of $[01]$ onto itself which leaves the endpoints of the segment fixed, is a strictly monotonic function. Hence (1.42) is incompatible with (1.41). This proves the assertion.

In the same way, a topological mapping of a circumference onto a Jordan curve induces a direction of circuit or sense on the latter by means of a given sense on the former.

More precisely, an oriented Jordan curve is a simple closed curve Φ on which every simple arc $\Phi_0 \subset \Phi$ has been assigned a definite sense in such a way that if $\Phi_0 \subset \Phi$, $\Phi_1 \subset \Phi$ are any two arcs on Φ and $\Phi_0 \subseteq \Phi_1$, then the sense assigned to Φ_0 coincides with that assigned to Φ_1.

An oriented Jordan curve will be called simply an oriented curve and will be denoted by $\Phi^{\rightarrow}$ if the curve itself is denoted by Φ. If a Jordan curve is given by several letters, it will be written as e.g., $\langle abcd \rangle$. An oriented curve will then be symbolized by $\langle abcd \rangle^{\rightarrow}$.

§2. The angle function of a continuous mapping of a segment into the plane. The index of a point relative to a closed path in the plane

§2.1. The functions $F_a(p, C, x)$ and $f(p, C, x_1, x_2)$. Let C be a continuous mapping of the segment $[e_0e_1] \equiv e_0 \leq x \leq e_1$ of the real line into the plane R^2. Let $p \in R^2$ be a point of the complement of $\Phi = C([e_0e_1])$.

Finally, let a be a point of the segment $[e_0e_1]$. Let us suppose that in the plane R^2 a definite convention for measuring an angle in the positive sense (e.g., counterclockwise) has been chosen.

The angle $C(a)pC(x)$ is defined only to within an integral multiple of 2π. However, it can be defined uniquely for all x, $e_0 \leq x \leq e_1$, by choosing a definite value of this angle for a fixed $x = x_0$ and requiring that it be a continuous function of x. Indeed, let the points

$$a_0 = e_0, a_1, \cdots, a_s = e_1$$

divide $[e_0e_1]$ into segments $[a_0a_1]$, $[a_1a_2]$, $\cdots$, $[a_{s-1}a_s]$ so small that C maps each of them into the interior of some acute angle with vertex p. If the value of the angle $C(a)pC(x)$ is defined for a fixed x_0, $a_i \leq x_0 \leq a_{i+1}$, it is then uniquely defined by continuity for all x, $a_i \leq x \leq a_{i+1}$; in particular, therefore, for $x = a_i$ and $x = a_{i+1}$. But the value of the angle $C(a)pC(x)$ for $x = a_{i+1}$ uniquely determines the values of this angle for all $x \in [a_{i+1}a_{i+2}]$, and then for all $x \in [a_{i+2}a_{i+3}]$, etc. In exactly the same

way, if the angle $C(a)pC(x)$ is defined for $x = a_i$, its value is uniquely determined by continuity on the segment $[a_i a_{i-1}]$, etc. Hence the angle $C(a)pC(x)$ is defined for all $x \in [e_0 e_1]$.

DEFINITION 2.11. The value of the angle $C(a)pC(x)$, uniquely defined for all x, $e_0 \leq x \leq e_1$, by the condition that $C(a)pC(a) = 0$, is denoted by $F_a(p, C, x)$.

REMARK 1. The function $F_a(p, C, x)$ is clearly continuous in each of x, a, and p. Furthermore, it is easy to see that it is a continuous function of all its arguments p, C, x, a simultaneously, i.e., for fixed x, a, p, C and for every $\epsilon > 0$, there exists a $\delta > 0$ such that $F_{a'}(p', C', x')$ is defined and

$$| F_a(p, C, x) - F_{a'}(p', C', x') | < \epsilon$$

for

$$| x' - x | < \delta, \qquad | a' - a | < \delta, \qquad \rho(p', p) < \delta, \qquad \rho(C', C) < \delta.$$

Here, in accordance with I, 7.3, $\rho(C', C) < \delta$ means that $\rho[C'(\xi), C(\xi)] < \delta$ for all ξ, $e_0 \leq \xi \leq e_1$.

REMARK 2. If a, a' are any two points of the segment $[e_0 e_1]$,

$$F_{a'}(p, C, x) = F_{a'}(p, C, a) + F_a(p, C, x). \tag{2.11}$$

Hence $F_a(p, C, x_2) - F_a(p, C, x_1)$ is independent of a.

DEFINITION 2.12. Let

$$f(p, C, x_1, x_2) = F_a(p, C, x_2) - F_a(p, C, x_1) \tag{2.12}$$

for any $a \in [e_0 e_1]$; $f(p, C, x_1, x_2)$ is also a continuous function of its arguments.

§2.2. The index of a point relative to a closed path. Let us now assume that the mapping C satisfies the condition

$$C(e_0) = C(e_1).$$

The mapping C is then said to define a "closed path" in the plane. The intuitive geometric meaning of this term is clear. A rigorous definition of a closed path would have to state the conditions under which two continuous mappings determine the same closed path. However, we shall not use this notion and will leave it imprecise.

If C defines a closed path, the function $f(p, C) = f(p, C, e_0, e_1)$ is equal to an integral multiple of 2π. Hence an integer $\omega(C, p) = (1/2\pi)f(p, C)$, which depends only on C and p, is defined for every point p in the complement of $C([e_0 e_1])$. The number $\omega(C, p)$ is called the *index of the point p relative to the mapping C* (or the index of p relative to the closed path defined by C).

It is clear that if $\omega(C, p)$ is defined for a point p, it is also defined for all points sufficiently near p. Furthermore, since $\omega(C, p)$ is continuous in p and assumes only integral values, it follows that

$$\omega(C, p') = \omega(C, p)$$

for all p' sufficiently near p. Hence

2.21. *The function $\omega(C, p)$ assumes the same value for all points p contained in one component of the open set $R^2 \setminus C([e_0e_1])$.*

Indeed, if p_1 and p_2 are in the same component Γ of $R^2 \setminus C([e_0e_1])$, they can be joined by a broken line in Γ. The function $\omega(C, p)$ is defined for all points of this broken line and, because it is continuous and integer-valued, is constant there. Therefore,

$$\omega(C, p_1) = \omega(C, p_2).$$

This proves the assertion.

It is obvious that if the set $C([e_0e_1])$ is contained in the interior of an angle α, $0 < \alpha < 2\pi$, with vertex p, then $|f(p, C, x_1, x_2)| < \alpha$ for all $x_1, x_2 \in [e_0e_1]$.

Hence

2.22. *If C is a continuous mapping of the segment $[e_0e_1]$ into the plane R^2 such that $C(e_0) = C(e_1)$ and $C([e_0e_1])$ is contained in one of the two half-planes into which R^2 is separated by some straight line, then $\omega(C, p) = 0$ for any p contained in the other half-plane.*

Let us make two important remarks.

2.23. Let C be a continuous mapping of the segment $[e_0e_1]$ onto a circumference K, (1–1) on the open interval (e_0e_1) and such that $C(e_0) = C(e_1)$. Let p be the center of K. Then $\omega(C, p) = +1$ or -1 *depending on whether, as x varies from e_0 to e_1 on the segment $[e_0e_1]$, the point $C(x)$ describes K in the positive or negative sense.*

By virtue of 2.21, the same assertion is also valid for an arbitrary point in the interior of K.

The circumference K can obviously be replaced in 2.23 by, e.g., a triangular contour.

2.231. We shall call a deformation C_θ, $0 \leq \theta \leq 1$, of the mapping $C = C_0$ *admissible* if the fixed point p never meets the set $C_\theta([e_0e_1])$ and

$$C_\theta(e_0) = C_\theta(e_1)$$

for all θ, $0 \leq \theta \leq 1$. Then $\omega(C_\theta, p)$ is defined for $0 \leq \theta \leq 1$ and, since $F_a(p, c, x)$ is continuous in C and $\omega(C_\theta, p)$ is integer-valued, $\omega(C_\theta, p)$ is constant, $0 \leq \theta \leq 1$. Hence

2.24. *The index $\omega(C, p)$ is invariant under admissible deformations of the mapping C.*

§2.3. The addition formula. Let $[e_0e_1]$, $[e_1e_2]$, $e_0 < e_1 < e_2$, be two closed intervals of the real line and let C_1, C_2 be continuous mappings of $[e_0e_1]$, $[e_1e_2]$, respectively, into R^2 such that

$$C_1(e_1) = C_2(e_1). \tag{2.31}$$

Let C denote the mapping of the segment $[e_0e_2]$ coinciding with C_1 on $[e_0e_1]$ and with C_2 on $[e_1e_2]$. Finally, let p be a point of R^2 in the complement of the set

$$C_1([e_0e_1]) \cup C_2([e_1e_2]) = C([e_0e_2]).$$

Then (2.1, Remark 2)

$$F_{e_0}(p, C, e_2) = F_{e_0}(p, C, e_1) + F_{e_1}(p, C, e_2). \tag{2.32}$$

Now let

$$C(e_0) = C(e_1) = C(e_2), \tag{2.33}$$

i.e.,

$$C_1(e_0) = C_1(e_1) = C_2(e_1) = C_2(e_2).$$

Then $\omega(C_1, p)$, $\omega(C_2, p)$, $\omega(C, p)$ are defined and it follows easily from (2.32) and the definition of these numbers that

$$\omega(C, p) = \omega(C_1, p) + \omega(C_2, p). \tag{2.34}$$

We shall need one application of (2.34) in the sequel.

$$a = C_1(0) = C_1(2) = C_1(4),$$
$$b = C_1(1) = C_1(3),$$
$$[acb]^{\rightarrow} = C_1([01]),$$
$$[bga]^{\rightarrow} = C_1([12]),$$
$$[agb]^{\rightarrow} = C_1([23]),$$
$$[bda]^{\rightarrow} = C_1([34]).$$

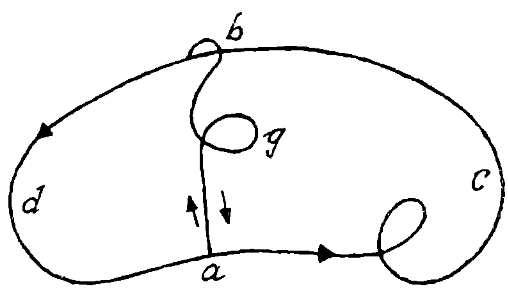


FIG. 12

Let C_1 be a continuous mapping of the segment $[04]$ of the real line into R^2 (Fig. 12) such that

$$\begin{aligned} &C_1(4) = C_1(2) = C_1(0), \\ &C_1(2 + x) = C_1(2 - x), \qquad 0 \le x \le 1. \end{aligned} \tag{2.35}$$

Hence

$$C_1(3) = C_1(1).$$

Let $p \in R^2$ be a point in the complement of $C_1([04])$. Denote by C^{02} and C^{24}, respectively, the mappings of the segments [02] and [24] defined by the mapping C_1. Finally, let C_0 be the mapping defined by

$$C_0(x) = C_1(x), \quad 0 \leq x \leq 1 \quad \text{and} \quad 3 \leq x \leq 4,$$
$$C_0(x) = C_1(1) = C_1(3), \qquad 1 \leq x \leq 3.$$

Then $\omega(C_1, p)$, $\omega(C^{02}, p)$, $\omega(C^{24}, p)$, $\omega(C_0, p)$ are defined and

$$\omega(C_0, p) = \omega(C^{02}, p) + \omega(C^{24}, p). \tag{2.36}$$

In fact, (2.34) implies that

$$\omega(C_1, p) = \omega(C^{02}; p) + \omega(C^{24}, p).$$

It remains to be shown that $\omega(C_0, p) = \omega(C_1, p)$.

However, the mapping C_0 is obtained from C_1 by an admissible deformation

$$\left.\begin{aligned} C_\theta(x) &= C_1(x), \\ C_\theta(1 + x) &= C_1(1 + \theta x), \\ C_\theta(2 + x) &= C_1[1 + \theta(1 - x)], \\ C_\theta(3 + x) &= C_1(3 + x); \end{aligned}\right\} \qquad \begin{aligned} 0 \leq x \leq 1, \\ 0 \leq \theta \leq 1. \end{aligned}$$

Hence $\omega(C_0, p) = \omega(C_1, p)$, since $\omega(C, p)$ is invariant under admissible deformations of C, and this at once implies (2.36).

§2.4. The index of a point relative to a Jordan curve. For the remainder of this section Φ will denote a plane Jordan curve. We shall call a continuous mapping C of the segment [01] of the real line onto a Jordan curve Φ which maps the open interval (01) topologically and which satisfies the condition

$$C(0) = C(1) = e' \in \Phi$$

a *simple mapping*.

A simple mapping of [01] defines an orientation of Φ: if $0 < a < b < 1$, the orientation induced on Φ by C shall be $\langle e'C(a)C(b)e' \rangle^{\rightarrow}$.

2.41. If $p \in R^2 \setminus \Phi$ and C, C' are two simple mappings of [01] onto a Jordan curve Φ which induce the same orientation on Φ, then

$$\omega(C, p) = \omega(C', p).$$

To prove this, it suffices, by 2.24, to show that C and C' can be deformed into each other by a deformation C_θ, $0 \leq \theta \leq 1$, such that

$$C_\theta([01]) = \Phi, \qquad C_\theta(0) = C_\theta(1) = e', \qquad 0 \leq \theta \leq 1. \tag{2.41}$$

It is enough to prove the last assertion for Φ a circumference of length 1. In that case, however, it follows from

2.42. Let C_0 be any simple mapping of [01] onto a circumference Φ of length 1. Let C_1 be the mapping such that

$$C_1(0) = C_0(0) = e',$$

and such that $C_1(x)$, $0 \leq x \leq 1$, is the endpoint of the arc of length x laid off on the circumference Φ from the point e' in the direction induced by the mapping C_0. Then C_0 can be deformed into C_1 by a deformation satisfying (2.41).

Indeed, for each θ, $0 \leq \theta \leq 1$, let

$$C_\theta(0) = C_\theta(1) = C_0(0).$$

Consider the arc $L_x = C_0(x)C_1(x)$ of Φ, $0 < x < 1$, which has $C_0(x)$ and $C_1(x)$ as its endpoints and which does not contain e'; define $C_\theta(x)$ as the point dividing the arc L_x in the ratio $\theta:(1 - \theta)$, i.e., such that

$$[C_0(x)C_\theta(x)]:[C_\theta(x)C_1(x)] = \theta:(1 - \theta).$$

Then C_θ is the required continuous deformation.

Exercise. Prove that two simple mappings C_0 and C_1 which induce opposite orientations on Φ cannot be deformed into each other.

Definition 2.4. Let $\Phi^{\rightarrow}$ be an oriented Jordan curve and let $p \in R^2 \setminus \Phi$; the number $\omega(C, p)$, where C is any simple mapping of [01] onto Φ inducing the given orientation on $\Phi^{\rightarrow}$, is called the *index* of the point p relative to the oriented Jordan curve $\Phi^{\rightarrow}$ and is denoted by $\omega(p, \Phi^{\rightarrow})$.

We shall require, finally, the following proposition:

2.43. Consider in R^2 three simple arcs with common endpoints a, b, no two of which have any points in common except for their endpoints. Denote these arcs by $[ab]_1$, $[ab]_2$, $[ab]_3$, respectively. From these we form three oriented Jordan curves

$$[ab]_1^{\rightarrow} \cup [ba]_2^{\rightarrow}, \qquad [ab]_1^{\rightarrow} \cup [ba]_3^{\rightarrow}, \qquad [ab]_2^{\rightarrow} \cup [ba]_3^{\rightarrow}.$$

Then for any point p not on any of the three arcs,

$$\omega(p, [ab]_1^{\rightarrow} \cup [ba]_2^{\rightarrow}) + \omega(p, [ab]_2^{\rightarrow} \cup [ba]_3^{\rightarrow}) = \omega(p, [ab]_1^{\rightarrow} \cup [ba]_3^{\rightarrow}). \tag{2.43}$$

This formula follows from (2.36) and the following almost obvious proposition:

2.431. Let C_0 be a continuous mapping of [01] onto a Jordan curve Φ such that C_0 maps $[0\frac{1}{3}]$ topologically onto some arc $\bar{\lambda}_1 = [e'e'_1] \subset \Phi$, the segment $[\frac{1}{3}\,\frac{2}{3}]$ onto e'_1, and finally the segment $[\frac{2}{3}1]$ topologically onto the arc $\bar{\lambda}_2 = [e'_1e'] \subset \Phi$ different from $\bar{\lambda}_1$. Then the mapping C_0 can be de-

formed into a simple mapping by means of a deformation C_θ satisfying (2.41).

It suffices to prove this proposition on the assumption that Φ is a circumference of length 1; then C_1 and C_θ are defined exactly as in the proof of 2.42.

§2.5. (This article will not be used in the proof of the Jordan theorem.) **The index of a point $p \in R^2$ relative to a continuous mapping of a circumference into $R^2 \setminus p$; degree of a continuous mapping of a circumference into a circumference.** Let S_β be an oriented circumference of length 1. Choose an arbitrary point a_β on S_β. We shall say that S_β is described in the positive direction if this corresponds to the given orientation of S_β.

Let C^β be a continuous mapping of S_β into R^2 and let $p \in R^2 \setminus C^\beta(S_\beta)$.

Let C_β^{01} be the continuous mapping of the segment $0 \leq x \leq 1$ onto the circumference S_β which assigns to each point x, $0 \leq x \leq 1$, the endpoint of the arc of length x laid off on S_β in the positive sense from the point a_β.

Then $C = C^\beta C_\beta^{01}$ is a mapping of the segment [01] into $R^2 \setminus p$ and $C(0) = C(1)$.

It is easy to see that the index of a point relative to the mapping C is independent of the choice of the intermediate mapping C_β^{01} (i.e., of the choice of the point a_β which completely determines C_β^{01}). Hence the index of the point p relative to the mapping $C = C^\beta C_\beta^{01}$ depends only on the mapping C^β (but also on the chosen orientation of S_β and the sense in which positive angles are measured in R^2). It is therefore natural to call this index the index of a point relative to the mapping C^β of the circumference S_β into $R^2 \setminus p$. Hence, by definition, let us put

$$\omega(C^\beta, p) = \omega(C, p).$$

Now let S_α be a circumference in R^2 with center p, oriented in the same sense in which positive angles are measured in R^2. If C_α^β is a continuous mapping of S_β into S_α, the index of the center p of S_α (just defined) relative to C_α^β is called the *degree* of the mapping C_α^β.

In this article we shall show that the degree of C_α^β completely characterizes the homotopy class of C_α^β (see I, 7.4). We shall begin by defining a normal mapping C_α^β of degree ω, where ω is an arbitrary integer.

We shall assume for simplicity that both circumferences S_α and S_β are of length 1.

Let us choose, once and for all, definite points $a_\beta \in S_\beta$ and $a_\alpha \in S_\alpha$. A mapping ν_α^β is called a *normal* mapping of degree ω of S_β into S_α if it assigns to each point $\xi \in S_\beta$ the point $\nu_\alpha^\beta(\xi) \in S_\alpha$ determined as follows. Take the arc $[a_\beta\xi]^\rightarrow$ on S_β in the positive sense. Lay off on S_α from the point a_α, in the positive sense if $\omega > 0$ and in the negative sense if $\omega < 0$, an arc $[a_\alpha\eta]^\rightarrow$ whose length is equal to the product of $|\omega|$ and the length

of the arc $[a_\alpha\xi]^{\rightarrow}$. The endpoint η of the arc $[a_\alpha\eta]^{\rightarrow} \subset S_\alpha$ is, by definition, the point $\nu_\alpha^\beta(\xi)$. The degree of the mapping ν_α^β is obviously ω; a normal mapping of degree 0 maps S_β onto the point a_α.

Our purpose will be achieved if we prove the following two propositions:

2.51. *Every continuous mapping C_α^β of S_β into S_α is homotopic to a normal mapping whose degree is the same as that of C_α^β.*

2.52. *Two homotopic continuous mappings of an oriented circumference S_β into an oriented circumference S_α have the same degree.*

Proof of 2.51. Let us first note that every continuous mapping of S_β into S_α is homotopic to a mapping C_α^β such that $C_\alpha^\beta(a_\beta) = a_\alpha$. (The homotopy is realized by a simple rotation through the appropriate angle.) Therefore, it suffices to prove Theorem 2.51 for a mapping C_α^β such that $C_\alpha^\beta(a_\beta) = a_\alpha$. We shall now proceed with the proof.

Let ω be the degree of the mapping C_α^β and consider the continuous function $F(C_\alpha^\beta, x)$ defined on the whole real line in the following way.

Let us set

$$F(C_\alpha^\beta, x) = F_0(p, C, x), \qquad 0 \le x \le 1,$$

where $C = C_\alpha^\beta C_\beta^{01}$ and p is the center of S_α (the mapping C_β^{01} is defined as above; the point $a_\beta \in S_\beta$ is at all times fixed). For all the remaining values of x let us define $F(C_\alpha^\beta, x)$ by means of the equation

$$F(C_\alpha^\beta, x + 1) = F(C_\alpha^\beta, x) + 2\pi\omega,$$

remembering that $\omega(C, p) = (1/2\pi)F_0(p, C, 1)$.

The function $F(C_\alpha^\beta, x)$ (for fixed a_β) is uniquely defined by the mapping C_α^β and is called the *angle function* (*turning function*) *corresponding to the mapping* C_α^β.

In general, we shall call any real continuous function $F(x)$ defined on the whole real line and satisfying the conditions

$$F(0) = 0,$$

$$F(x + 1) = F(x) + 2\pi\gamma,$$

where γ is a fixed integer, an *angle function.*

2.511. *Every angle function $F(x)$ is the angle function corresponding to some continuous mapping C_α^β of the circumference S_β into S_α such that $C_\alpha^\beta(a_\beta) = a_\alpha$.*

Indeed, let us assign to each point $\xi = C_\beta^{01}x$ of S_β the point η of S_α whose polar angle is $F(x)$ (measured from the point a_α). Denote the point η by $C_\alpha^\beta(\xi)$. It follows easily from the definition of an angle function that C_α^β is a single-valued continuous mapping of S_β into S_α.

$F(x)$ is obviously the angle function corresponding to C_α^β. Since, on

the other hand, the image of the point $\xi = C_\beta^{01}x \in S_\beta$ under the mapping C_α^β was uniquely defined by its polar angle $F(C_\alpha^\beta, x)$, there exists just one continuous mapping C_α^β [satisfying the condition $C_\alpha^\beta(a_\beta) = a_\alpha$], to which the given angle function

$$F(x) = F(C_\alpha^\beta, x)$$

corresponds.

We have thus established a (1–1) correspondence between the continuous functions C_α^β of S_β into S_α, for which $C_\alpha^\beta(a_\beta) = a_\alpha$, and the angle functions.

Let us note, finally, the following proposition, whose proof may be left to the reader:

2.512. If $F_\theta(x)$, $0 \leq \theta \leq 1$, is a family of angle functions, continuous in the parameter θ, then the corresponding mappings C_α^β are also continuous in θ.

The proof of Theorem 2.51 can now be concluded without any difficulty.

Let us set

$$F_\theta(x) = (1 - \theta)F(x) + 2\pi\theta\omega x, \qquad 0 \leq \theta \leq 1,$$

where $F(x) = F_0(x)$ is the angle function corresponding to the mapping C_α^β of degree ω. We shall prove that $F_\theta(x)$ is an angle function. Since

$$F_\theta(0) = (1 - \theta)F(0) = 0,$$

it suffices to prove that

$$F_\theta(x + 1) - F_\theta(x) = 2\pi\omega.$$

But

$$F_\theta(x + 1) - F_\theta(x) = (1 - \theta)[F(x + 1) - F(x)] + 2\pi\theta\omega.$$

Since $F(x)$ is the angle function *corresponding to the mapping C_α^β of degree ω*,

$$F(x + 1) - F(x) = 2\pi\omega,$$

so that

$$F_\theta(x + 1) - F_\theta(x) = (1 - \theta)2\pi\omega + 2\pi\theta\omega = 2\pi\omega.$$

Therefore, all the $F_\theta(x)$ are angle functions, continuous in θ and deforming $F_0(x)$ into the angle function

$$F_1(x) = 2\pi\omega x,$$

which obviously corresponds to a normal mapping of degree ω. This, by virtue of 2.512, completes the proof of Theorem 2.51.

Proof of 2.52. Let ${}_0C_\alpha{}^\beta$ and ${}_1C_\alpha{}^\beta$ be two homotopic mappings. It is required to prove that they have the same degree.

Let the deformation ${}_\theta C_\alpha{}^\beta$, $0 \leq \theta \leq 1$, carry ${}_0C_\alpha{}^\beta$ into ${}_1C_\alpha{}^\beta$.

Let $C_\beta{}^{01}$ be the mapping of the segment [01] onto S_β defined at the beginning of this article (see p. 52). Then ${}_\theta C_\alpha{}^\beta C_\beta{}^{01}$ is an admissible deformation of ${}_0C_\alpha{}^\beta C_\beta{}^{01}$ into ${}_1C_\alpha{}^\beta C_\beta{}^{01}$, so that, by 2.24, the index of the center p of S_α is the same with respect to both mappings ${}_0C_\alpha{}^\beta C_\beta{}^{01}$ and ${}_1C_\alpha{}^\beta C_\beta{}^{01}$. But this also means that the mappings ${}_0C_\alpha{}^\beta$ and ${}_1C_\alpha{}^\beta$ (of S_β into S_α) have the same degree. This proves Theorem 2.52.

Theorems 2.51, 2.52, and the existence of a normal mapping of degree ω for any integer ω may be combined in one proposition.

2.5. *There exist continuous mappings of arbitrary degree ω of an oriented circumference S_β into an oriented circumference S_α ; two continuous mappings of S_β into S_α are homotopic if, and only if, they have the same degree.*

REMARK 1. Let C be any continuous mapping of a circumference S_β into $R^2 \setminus p$, $S_\alpha \subset R^2$ a circumference with center p, and C_α the central projection (with center p) of the set $R^2 \setminus p$ into S_α.

The index of the point p relative to the mapping C is equal to the degree of the mapping $C_\alpha C$ of S_β into S_α. (This assertion follows from 2.24.)

REMARK 2. The methods of this section admit of very diverse applications—to analysis, to the theory of functions of a complex variable, etc. Thus, e.g., the fundamental theorem of algebra can be proved, literally in two words, by means of the notion of the index of a point relative to a continuous mapping.

§3. Theorem: A simple arc does not disconnect the plane

DEFINITION 3.1. Let Φ be a closed subset of the plane R^2; we shall say that Φ does not separate two points p and q of the open set $R^2 \setminus \Phi$, if p and q are in one component of $R^2 \setminus \Phi$, i.e., if they can be joined by a simple broken line in $R^2 \setminus \Phi$.

To prove the theorem stated at the head of this section we note first:

3.2. Let $\Phi = [ab]$ be a simple arc in R^2 and $c \neq b$ a point of Φ; further, let p, q be points of $\Gamma = R^2 \setminus \Phi$. If the arc $[ac] \subset \Phi$ does not separate p and q (if $c = a$, the arc $[ac]$ becomes the point a), there exists a point c' on the arc $(cb) \subset \Phi$ such that the arc $[ac']$ does not separate p and q.

Indeed, let $[pq]$ be a broken line joining p and q in the complement of $[ac]$. Then

$$\rho([ac], [pq]) = \epsilon > 0.$$

Let us take a point c', $c' \neq c$, on the arc $[cb]$ so near to c that

$$\delta([cc']) < \epsilon.$$

Clearly, $[cc']$, and therefore $[ac']$ also, has no points in common with $[pq]$. This proves the assertion.

3.3. *If p and q are not separated by any subarc $[ac]$ of the simple arc $[ab]$, $c \neq b$, then the arc $[ab]$ does not separate p and q.*

Before proving this proposition, we shall deduce the theorem stated at the head of this section from 3.2 and 3.3.

Let us suppose that the simple arc $\Phi = [ab]$ disconnects the plane. Then there are two points p and q which are separated by $[ab]$.

The set consisting of the single point a obviously does not separate p and q. Therefore, by 3.2, there is a subarc $[aa']$ of $[ab]$ which does not separate p and q. Let c be the least upper bound (on the oriented arc $[ab]^{\rightarrow}$) of the set of those points $a' \in [ab]$, for which the arc $[aa'] \subseteq [ab]$ does not separate p and q.

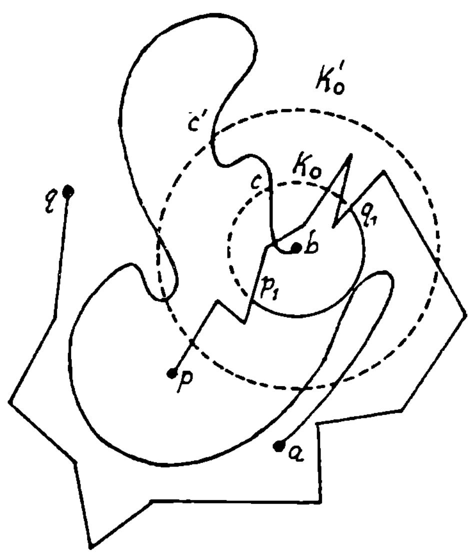


Fig. 13

By 3.3, the arc $[ac]$ does not separate p and q. In virtue of 3.2, c is identical with b, i.e., $[ab]$ does not separate p and q. Hence it remains to prove 3.3.

Let K' be a circle with center b such that a, p, q are in the exterior of K'. Let c' be the last point of $[ab] \cap K'_0$ (on $[ab]^{\rightarrow}$), where K'_0 is the circumference of the circle K'. The arc $[ac'] \subset [ab]$ has a positive distance from b. Let K be a circle with center b and radius less than $\rho(b, [ac'])$; denote the circumference of this circle by K_0. Hence the first point (on $[ab]^{\rightarrow}$) of $K_0 \cap [ab]$ comes, on $[ab]^{\rightarrow}$, after the last point of $K'_0 \cap [ab]$, *so that every point which follows a point of $K_0 \cap [ab]$ on $[ab]^{\rightarrow}$ is in the interior of K'.*

We shall now prove 3.3 first for the special case that $[ab] \cap K_0$ consists of a single point c.

Since $[ac]$, by hypothesis, does not separate p and q, there exists a broken line $[pq]$ not meeting $[ac]$ (Fig. 13). Let p_1 be the first, and q_1 the last, point of $[pq] \cap K_0$ on $[pq]^{\rightarrow}$. Both p_1 and q_1 are different from c and can therefore be joined by an arc $[p_1q_1]$ of the circumference K_0 not containing the point c. The arc $[pp_1] \cup [p_1q_1] \cup [q_1q]$ (where $[pp_1] \subset [pq]$, $[p_1q_1] \subset K_0$, $[q_1q] \subset [pq]$) joins p to q and has no points in common with $[ab]$.

Let us now pass to the case in which $[ab] \cap K_0$ consists of more than one point (Fig. 14).

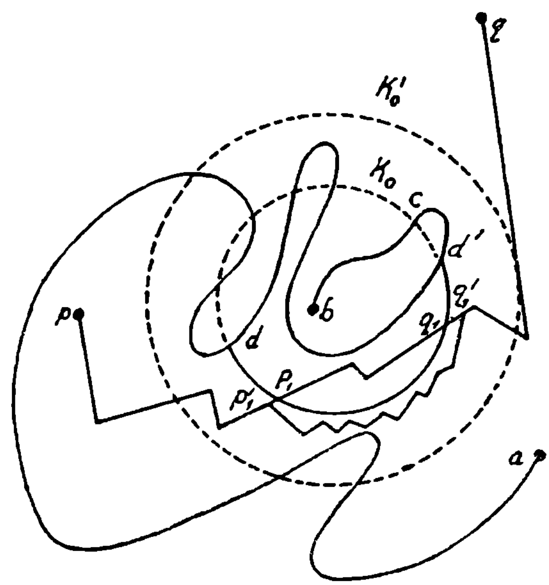


FIG. 14

Let us denote by $\lambda^{\rightarrow} = (dd')^{\rightarrow}$ any open arc of K_0 in the complement of $[ab] \cap K_0$ but with its endpoints in $[ab] \cap K_0$ and oriented in the positive sense (i.e., in the sense corresponding to a counterclockwise circuit of K_0).

Let $\lambda'^{\rightarrow} = (d'd)^{\rightarrow}$ be the oriented arc of K_0 complementary to the arc λ (hence the orientations of λ and λ' correspond to a counterclockwise circuit of K_0).

Denote by C_λ^{01} a topological mapping of the segment $[01]$ of the real line onto the simple arc $\bar{\lambda}$, by $C_{d'd}^{12}$ a topological mapping of $[12]$ onto the oriented arc $[d'd]^{\rightarrow} \subset [ab]$, and by $C_{dd'}^{23}$ the topological mapping of $[23]$ onto the oriented arc $[dd']^{\rightarrow} \subset [ab]$ defined by the equation

$$C_{dd'}^{23}(2 + x) = C_{d'd}^{12}(2 - x), \qquad 0 \leq x \leq 1.$$

Finally, let $C_{\lambda'}^{34}$ be a topological mapping of $[34]$ onto $\bar{\lambda}'$.

The mappings C_λ^{01}, $C_{d'd}^{12}$, $C_{dd'}^{23}$, $C_{\lambda'}^{34}$ combined yield a mapping C of $[04]$ satisfying (2.35); the corresponding mapping C_0 is obviously a mapping of $[04]^{\rightarrow}$ onto K_0 oriented counterclockwise. The mapping C^{02} is a

mapping of [02] onto the "closed path" consisting of the arc $\lambda^{\to}$ and the arc $[d'd]^{\to} \subset [ab]$, and C^{24} is a mapping of [24] onto the closed path consisting of the arcs $[dd']^{\to} \subset [ab]$ and $\lambda'^{\to}$. By (2.36),

$$\omega(C_0, x) = \omega(C^{02}, x) + \omega(C^{24}, x), \tag{3.31}$$

i.e.,

$$\omega(C^{02}, x) = \omega(C_0, x) - \omega(C^{24}, x), \tag{3.31'}$$

for any point x in the complement of $K_0 \cup [d'd]$ (where $[d'd] \subset [ab]$).

Every point of the arc $[d'd] \subset [ab]$ follows on $[ab]^{\to}$ one of the points d', d of this arc, i.e., it follows some point of the set $K_0 \cap [ab]$. Hence $[d'd] \subset [ab]$ is in the interior of the circle K', so that $C^{02}([02]) = \lambda \cup [d'd]$ is also contained in the interior of K'. On the other hand, p and q are in the exterior of K'. Hence, by 2.22,

$$\omega(C^{02}, p) = 0, \qquad \omega(C^{02}, q) = 0. \tag{3.32}$$

Let c be the last point of $[ab] \cap K_0$ on $[ab]^{\to}$.

Let us say that a broken line $[pq]$ joining p and q is *admissible* if it satisfies the following conditions:

1. $[pq] \cap [ac] = 0$.
2. K_0 does not contain any vertex of $[pq]$.
3. No link of the broken line $[pq]$ is tangent to K_0.

An admissible broken line $[pq]$ exists. Indeed, by hypothesis, p and q can be joined by a broken line having no point in common with the arc $[ac]$. Conditions 2 and 3 can be achieved by an arbitrarily small modification of this broken line (Fig. 15).

If the admissible broken line $[pq]$ has points in common with $[ab]$, all these points lie on the arc $[cb]^{\to} \subset [ab]$, i.e., in the interior of the circle K. Hence, *if an admissible broken line $[pq]$ has points in common with $[ab]$, it necessarily intersects the circumference K_0 and, moreover, none of the points of intersection of $[pq]$ with K_0 are on $[ab]$.*

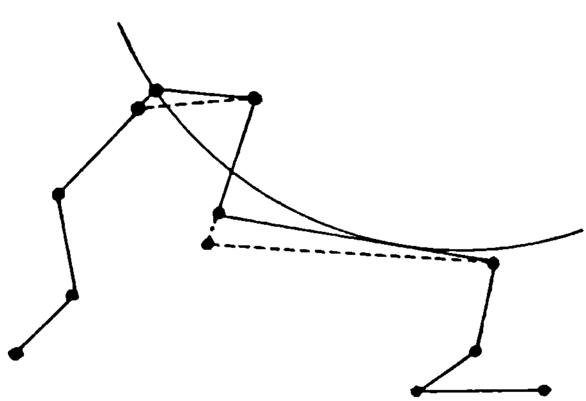


FIG. 15

3.31. *If the admissible broken line* $[pq]$ *intersects* K_0, *it can be replaced by an admissible broken line* $[pq]'$, *the number of whose points of intersection with* K_0 *is less than that of* $[pq]$ *with* K_0.

Proof of 3.31. Among the points of intersection of the oriented broken line $[pq]^{\rightarrow}$ with K_0 let us distinguish between "entrance points" (points at which $[pq]^{\rightarrow}$ enters the interior of the circle K) and "exit points" (points at which $[pq]^{\rightarrow}$ leaves the circle K).

Let p_1 be the first entrance point on $[pq]^{\rightarrow}$. It lies on some interval λ contiguous to $K_0 \cap [ab]$ (Fig. 14). Since the endpoints d and d' of λ are points of intersection of K_0 with $[ab]$ and c is the last point of $K_0 \cap [ab]$ on $[ab]^{\rightarrow}$, it follows that every point of $(dd')^{\rightarrow} \subset [ab]$ precedes c on $[ab]^{\rightarrow}$. Hence *the broken line* $[pq]$ *has no points in common with* $[dd'] \subset [ab]$. Let x be a variable point on $[pq]^{\rightarrow}$. Since $p_1 \notin C^{24}([24]) = [dd'] \cup \lambda'$, $\omega(C^{24}, x)$ does not change as x passes through p_1. The function $\omega(C_0, x)$, on the other hand, obviously jumps by $+1$ as x passes through p_1 or, in general, through any entrance point situated on λ. In consequence of (3.31′), $\omega(C^{02}, x)$ also jumps by $+1$ as x passes through an entrance point situated on λ.

Since $\omega(C^{02}, q) = 0$, a point moving on $[pq]^{\rightarrow}$ from p_1 to q must necessarily meet points of intersection of $[pq]$ with $C^{02}([02]) = [d'd] \cup \lambda$, i.e., points of intersection of $[pq]$ with λ; among these, in virtue of what has just been proved, there must be exit points. Let q_1 be the last exit point on $[pq]^{\rightarrow}$ contained in λ. Let us take two points p'_1 and q'_1 on $[pq]^{\rightarrow}$ very near to p_1 and q_1, respectively, and such that p'_1 precedes p_1 on $[pq]^{\rightarrow}$ and q'_1 follows q_1 on $[pq]^{\rightarrow}$. If p'_1 and q'_1 are taken sufficiently close to p_1 and q_1, they will both obviously be in the exterior of K. Since $[p_1q_1] \subset \lambda$ has a positive distance from $[ab]$, then (still on the assumption that p'_1 and q'_1 have been taken sufficiently near p_1 and q_1) there exists a broken line $[p'_1q'_1]$ contained entirely in the exterior of K, passing very near to $[p_1q_1]$, and having no points in common with $[ab]$.

Let us now take the broken line

$$[pp'_1] \cup [p'_1q'_1] \cup [q'_1q],$$

consisting of the piece $[pp'_1]$ of $[pq]$, of the broken line $[p'_1q'_1]$ just constructed, and of the piece $[q'_1q]$ of $[pq]$.

The resulting broken line $[pp'_1] \cup [p'_1q'_1] \cup [q'_1q]$ is obviously admissible; the number of its points of intersection with K_0 is at least two less than the number of points of intersection of the initial broken line $[pq]$ with K_0.

This proves 3.31 and also all of Theorem 3.3, since application of 3.31 a finite number of times to any admissible broken line $[pq]$ yields, in the end, an admissible broken line joining p and q and having no point of intersection with K_0. Such a broken line joining p and q does not have, as we know, points in common with $[ab]$, which was to be proved.

§4. Proof of the Jordan theorem

§4.1. Fundamental auxiliary construction. Let s be an arbitrary point of the plane in the complement of the curve Φ. The point s is to be regarded as fixed during the entire course of this article.

From the point s let us draw two distinct rays sa' and sb', where a', b' are points of the curve. (Rays will be designated by two letters not enclosed in brackets. The first letter denotes the initial point of the ray, the second another point of the ray.) Define the sense on each of these rays from s to a' and from s to b', respectively. Then let a be the first point of intersection of sa' with Φ and let b be the first point of intersection of sb' with Φ.

Let us take a definite orientation on Φ; we shall agree that this is the positive sense on Φ. Of the two arcs into which Φ is separated by the points a and b, denote the one on which the sense from a to b is positive by Λ_1, and the other by Λ_2.

Now let $\Phi_1^{\rightarrow}$ be the oriented Jordan curve consisting of the arc $\Lambda_1^{\rightarrow} = (ab)^{\rightarrow}$ and of the broken line $[bsa]^{\rightarrow}$, and let $\Phi_2^{\rightarrow}$ be the oriented Jordan curve consisting of the arc $\Lambda_2^{\rightarrow} = (ba)^{\rightarrow}$ and of the broken line $[asb]^{\rightarrow}$. The conditions of 2.43 are satisfied, so that

$$(4.11) \qquad \omega(p, \Phi_1^{\rightarrow}) + \omega(p, \Phi_2^{\rightarrow}) = \omega(p, \Phi^{\rightarrow})$$

for any point p in the complement of $\Phi \cup [asb]$.

The function $\omega(x, \Phi_1^{\rightarrow})$ is defined and constant for $x \in \Lambda_2$. Denote this constant value by ω_1. In exactly the same way, $\omega(x, \Phi_2^{\rightarrow})$ is constant for $x \in \Lambda_1$; denote its value for $x \in \Lambda_1$ by ω_2. We shall prove the following fundamental lemma:

4.11. *If at least one of the inequalities*

$$(4.12) \qquad \omega(p, \Phi_1^{\rightarrow}) \neq \omega_1, \qquad \omega(p, \Phi_2^{\rightarrow}) \neq \omega_2,$$

holds for a point $p \in R^2 \setminus \Phi$, then p and s can be joined by a broken line in $R^2 \setminus \Phi$.

Proof. Suppose, e.g., that

$$(4.121) \qquad \omega(p, \Phi_1^{\rightarrow}) \neq \omega_1 .$$

In consequence of the fundamental theorem of §3, there exists a broken line $[ps]$ in $R^2 \setminus \bar{\Lambda}_1$. Let p_1 be the first point on $[ps]^{\rightarrow}$ contained in $[asb]$. Then there is no point of Φ_1 on the piece $(pp_1)^{\rightarrow}$ of $[ps]$, so that $\omega(x, \Phi_1^{\rightarrow})$ is defined and constant for all $x \in (pp_1)$. Since, by (4.121), the value of $\omega(x, \Phi_1^{\rightarrow})$ for all $x \in [pp_1]$ is different from ω_1, i.e., from the value of the same function on Λ_2, it follows that no point of $[pp_1]$ is a point of Λ_2. In other words, the broken line $[pp_1]$, which does not intersect the arc $\bar{\Lambda}_1$, does not intersect Λ_2 either. This means that it does not intersect Φ. Since $p_1 \in [asb]$ and is different from the points a, b, it can be connected with s

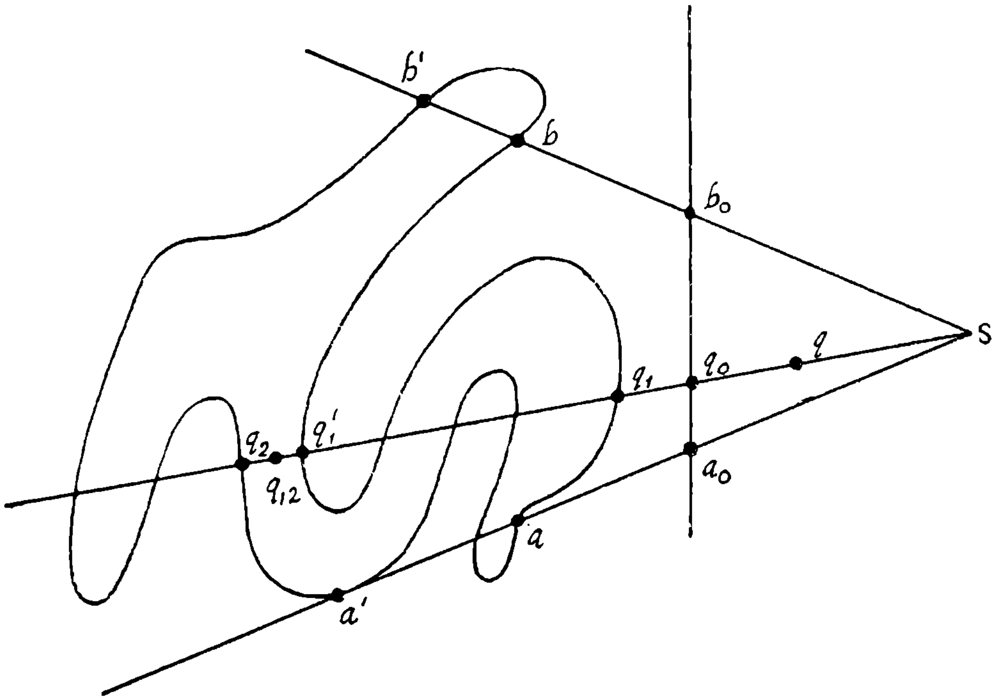


FIG. 16

by the segment $[p_1s]$ contained in $[asb]$. The broken line $[pp_1] \cup [p_1s]$ joins p and s in $R^2 \setminus \Phi$. This proves the lemma.

§4.2. The case $\omega(s, \Phi^{\rightarrow}) = 0$; the set $\Gamma = R^2 \setminus \Phi$ consists of at least two components. Let us draw a straight line D which does not intersect Φ, so that Φ is contained in one of the two half-planes into which R^2 is separated by D; let s be an arbitrary point of the half-plane which contains no points of Φ. The point s will remain fixed for the remainder of this article (Fig. 16).

By 2.22,

$$\omega(s, \Phi^{\rightarrow}) = 0. \tag{4.21}$$

Under the conditions assumed for s and using the notation of the preceding article, we shall prove

4.21. *One of the numbers ω_1, ω_2 is equal to zero, and the other is ± 1.*

Proof. Let a_0 and b_0 be the points of intersection of the straight line D with the rays sa and sb, respectively. Let us draw from s a ray sq_0 intersecting D in some point q_0 between a_0 and b_0. Let q be a point on sq_0 between s and q_0. Denote by $\Phi_3^{\rightarrow}$ the oriented Jordan curve consisting of the arc $\Lambda_1^{\rightarrow} = (ab)^{\rightarrow} \subset \Phi$ and the broken line $[bb_0a_0a]^{\rightarrow}$. Finally, let us denote by $\Delta^{\rightarrow}$ the triangular contour $\langle a_0b_0s\rangle$ oriented in the sense $\langle a_0b_0s\rangle^{\rightarrow}$. Then, by Theorem 2.43,

$$\omega(q, \Phi_1^{\rightarrow}) = \omega(q, \Phi_3^{\rightarrow}) + \omega(q, \Delta^{\rightarrow}).$$

Since

$$\omega(q, \Delta^{\rightarrow}) = \pm 1, \qquad \omega(q, \Phi_3^{\rightarrow}) = 0,$$

it follows that

$$(4.211) \qquad \omega(q, \Phi_1^{\rightarrow}) = \pm 1.$$

Similarly,

$$(4.212) \qquad \omega(q, \Phi_2^{\rightarrow}) = \pm 1.$$

As x goes to infinity along the ray sq, $\omega(x, \Phi_1^{\rightarrow})$ and $\omega(x, \Phi_2^{\rightarrow})$ will eventually become, and ever after remain, 0:

$$\omega(x, \Phi_1^{\rightarrow}) = 0, \qquad \omega(x, \Phi_2^{\rightarrow}) = 0.$$

Therefore the ray sq intersects both arcs Λ_1 and Λ_2.

Let Λ_1 be the first of the two arcs Λ_1 and Λ_2 which is met by sq as x goes to infinity on sq. We shall show that then

$$(4.213) \qquad \omega_2 = \pm 1, \qquad \omega_1 = 0.$$

(This will also prove 4.21. If Λ_2 had been the first arc met by sq, we would have had $\omega_2 = 0$, $\omega_1 = \pm 1$.)

Let q_1 be the first point of the ray sq on the arc Λ_1; the segment $[qq_1]$ has no points in common with Φ_2, so that $\omega(x, \Phi_2^{\rightarrow})$ is constant for points of this segment. Since $\omega(x, \Phi_2^{\rightarrow}) = \pm 1$, $\omega(q_1, \Phi_2^{\rightarrow}) = \pm 1$. Since $q_1 \in \Lambda_1$,

$$(4.214) \qquad \omega_2 = \pm 1.$$

In particular, if q'_1 is the last point of sq on Λ_1, then

$$(4.215) \qquad \omega(q'_1, \Phi_2^{\rightarrow}) = \pm 1.$$

Because of this and the fact that $\omega(x, \Phi_2^{\rightarrow}) = 0$ for all points of sq sufficiently far from the curve Φ, it follows that a point moving along sq from q'_1 to infinity will certainly meet the curve Φ_2, and consequently the arc Λ_2. Let q_2 be the first point of sq after q'_1 on Λ_2. Since q'_1 is the last point of Λ_1 on sq, $\omega(x, \Phi_1^{\rightarrow})$ is constant on the entire extent of sq from q'_1 to infinity, and this constant value of $\omega(x, \Phi_1^{\rightarrow})$ is zero. Hence

$$\omega(q_2, \Phi_1^{\rightarrow}) = 0.$$

Since $q_2 \in \Lambda_2$, the last equation implies that:

$$(4.216) \qquad \omega_1 = 0.$$

This proves (4.213) and thus 4.21.

REMARK. Since $\omega(x, \Phi_1^{\rightarrow}) = 0$ for all points x of sq after q'_1, it follows,

in particular, that

$$\omega(q_{12}, \Phi_1^{\rightarrow}) = 0 \tag{4.217}$$

for any point q_{12} of sq between q'_1 and q_2.

We shall now show that the curve Φ separates the plane into at least two components. Since $\omega(p, \Phi^{\rightarrow}) = 0$ for all $p \in R^2 \setminus \Phi$ sufficiently far from Φ, it suffices, by Theorem 2.21, to exhibit a point $p \in R^2 \setminus \Phi$ for which $\omega(p, \Phi^{\rightarrow}) \neq 0$. We shall show that, e.g., q_{12} is such a point. Indeed, since the segment $[q'_1 q_{12}]$ has no points in common with Λ_2, and therefore with Φ_2,

$$\omega(q_{12}, \Phi_2^{\rightarrow}) = \omega(q'_1, \Phi_2^{\rightarrow}).$$

Consequently, by (4.215),

$$\omega(q_{12}, \Phi_2^{\rightarrow}) = \pm 1. \tag{4.218}$$

(4.218), (4.217), and (4.11) imply

$$\omega(q_{12}, \Phi^{\rightarrow}) = \pm 1, \qquad \text{q.e.d.} \tag{4.219}$$

We shall deduce the following proposition from 4.11 and 4.21:

4.22. *Every $p \in R^2 \setminus \Phi$ such that*

$$\omega(p, \Phi^{\rightarrow}) = 0 \tag{4.220}$$

can be joined to s by a broken line in $R^2 \setminus \Phi$.

Proof. (4.11) and (4.220) imply that

$$\omega(p, \Phi_1^{\rightarrow}) + \omega(p, \Phi_2^{\rightarrow}) = 0.$$

This and 4.21 imply that at least one of the inequalities

$$\omega(p, \Phi_1^{\rightarrow}) \neq \omega_1, \qquad \omega(p, \Phi_2^{\rightarrow}) \neq \omega_2$$

must hold. Therefore 4.22 follows from 4.11.

Since $\omega(x, \Phi^{\rightarrow})$ is constant for all $x \in [sp] \subset R^2 \setminus \Phi$, the set of all points satisfying (4.220) is connected; it is, moreover, obviously open and hence is a domain $\Gamma_0 \subset R^2 \setminus \Phi$. It follows from 2.21, furthermore, that Γ_0 is a component of $R^2 \setminus \Phi$.

§4.3. Conclusion of the proof of the Jordan theorem. We have therefore proved:

4.31. *The set $R^2 \setminus \Phi$ has at least two components and, moreover, the set of all $p \in R^2 \setminus \Phi$ such that*

$$\omega(p, \Phi^{\rightarrow}) = 0 \tag{4.31}$$

is a component of $R^2 \setminus \Phi$.

It remains to be proved that:

4.32. *The set of all* $p \in R^2 \setminus \Phi$ *for which*

$$\omega(p, \Phi^{\rightarrow}) \neq 0 \tag{4.32}$$

is the second component of $R^2 \setminus \Phi$.

To prove this, it suffices to prove in turn that:

4.320. *If* $s, p \in R^2 \setminus \Phi$ *satisfy the conditions*

$$\omega(s, \Phi^{\rightarrow}) \neq 0, \qquad \omega(p, \Phi^{\rightarrow}) \neq 0, \tag{4.320}$$

there exists a broken line $[sp] \subset R^2 \setminus \Phi$ *joining* p *and* s.

Proof of 4.320. We shall regard the point s as fixed for the rest of this article, and make the same construction as in 4.1. Let us now show that, under the present conditions and with the notation of 4.1,

$$\omega_1 = \omega_2 = 0. \tag{4.321}$$

Let Q be any circle containing the curves Φ, Φ_1, Φ_2 in its interior and let s_0 be any point in the exterior of Q. Then

$$\omega(s_0, \Phi^{\rightarrow}) = 0, \tag{4.322}$$

$$\omega(s_0, \Phi_1^{\rightarrow}) = 0, \tag{4.323}$$

$$\omega(s_0, \Phi_2^{\rightarrow}) = 0. \tag{4.324}$$

Let $[s_0s]$ be a broken line joining s_0 and s in $R^2 \setminus \bar{\Lambda}_1$ (such a broken line exists according to §3). Since $\omega(x, \Phi^{\rightarrow})$ assumes different values for $x = s_0$ and $x = s$, $[s_0s]$ intersects Φ and consequently (since it does not intersect $\bar{\Lambda}_1$) has points in common with Λ_2. Let c_2 be the first point of Λ_2 on $[s_0s]^{\rightarrow}$. Then $\omega(x, \Phi^{\rightarrow})$ is constant on $(s_0c_2) \subset [s_0s]$ and is equal to $\omega(s_0, \Phi^{\rightarrow})$. On the other hand, $\omega(x, \Phi^{\rightarrow})$ is also constant on $(sa)^{\rightarrow}$ and $(sb)^{\rightarrow}$ and takes the value $\omega(s, \Phi^{\rightarrow}) \neq 0$. It follows that $[s_0c_2]$ has no points in common with $[asb]$ and, therefore, none with the curve Φ_1. Therefore, $\omega(x, \Phi_1^{\rightarrow})$ is constant on (s_0c_2) and, by (4.323), is equal to zero. Since $c_2 \in \Lambda_2$, $\omega_1 = 0$. It can be shown in exactly the same way that $\omega_2 = 0$.

4.320 follows from (4.321) without any difficulty. Indeed, in consequence of (4.32), (4.321), and (4.11), at least one of the inequalities

$$\omega(p, \Phi_1^{\rightarrow}) \neq \omega_1, \qquad \omega(p, \Phi^{\rightarrow}) \neq \omega_2$$

must hold. Then 4.320 follows from the fundamental lemma 4.11. This completes the proof of the Jordan theorem.

4.31, 4.32, and (4.218) imply

4.3. *If* Φ *is a Jordan curve and* Γ_0, Γ *are the two domains into which it separates the plane, then* $\omega(p, \Phi^{\rightarrow})$ *assumes the same value for every point* p *of each of these domains. This value is zero for the exterior (i.e., unbounded) domain and* ± 1 *for the interior (i.e., bounded) domain.*

Chapter III

SURFACES

In this chapter we shall classify topologically the simplest two-dimensional figures, the closed surfaces.

The exposition of the fundamental properties of these surfaces is preceded by a brief survey of the topology of *elementary curves* (§1).

Closed surfaces, as well as polyhedra and two-dimensional complexes—triangulations and their subcomplexes—are defined in §2. §2 also treats the combinatorial (Euler) characteristic of triangulations of closed surfaces and introduces the notion of a two-dimensional skeleton complex, by means of which the identification of the elements of a complex is rigorously defined.

§3 is devoted to a rigorous presentation of all definitions having to do with cuts and identifications, and to a proof of the relevant elementary theorems on which the further development (§§5–7) rests to a considerable extent.

§4 and §5 treat the orientability and connectivity of surfaces. The normal forms of simple (schlichtartige) surfaces and of closed surfaces are derived in §6 and §7, respectively. The method used here to derive the normal forms of closed surfaces is due basically to Alexander (see Alexander [a]).

The need of acquainting the reader with the elementary and geometrically intuitive material has led to some looseness in the exposition of this chapter. In contrast with the rest of the book, a great deal of the material in Chapter III depends on certain invariance theorems, which are not proved until Chapters V and X and are merely stated in the present chapter.

This procedure seems to me legitimate, since the proofs of these invariance theorems given subsequently are indepenent of the results of Chapter III. Otherwise, the exposition of the topology of surfaces would have to be postponed until much later, or burdened with a proof of invariance for the special case of two dimensions.

Still greater liberty is taken in the proof of the fundamental theorem of the theory of surfaces itself and in the constructions immediately preceding it. For the rest, since all the fundamental concepts pertaining to cuts and identifications have, I hope, been presented with impeccable rigor, I thought it important to emphasize the *applications of these ideas* and to leave to the reader the details of the proofs as exercises. These are routine (like calculations in classical analysis) because of the completeness of the definitions; to have included them in the book would only have served to encumber the exposition and diminish its clarity.

§1. Elementary curves and 1-complexes

§1.1. Elementary curves and their subdivision into arcs.

DEFINITION 1.11. A compactum which is the union of a finite number of simple arcs having by pairs at most a finite number of points in common is called an *elementary curve*. A point p of an elementary curve Φ is said to be *regular* if it has a neighborhood relative to Φ which is an open arc, i.e., a homeomorph of the straight line. (An open arc is a simple arc without its two endpoints; an open arc is homeomorphic to the real line, but not every set homeomorphic to the line is an open arc: for example, the graph of the function $y = \sin 1/x$, $0 < x < 1/\pi$, is homeomorphic to the line, but is not an open arc. However, every subset of an elementary curve homeomorphic to the line is either an open arc or a simple closed curve with one point deleted.) In the contrary case, the point p is said to be *singular*. Singular points are, in turn, divided into *branch points* and *endpoints*. A branch point is the common endpoint of three or more simple arcs of Φ which have no points in common except for p. A point $p \in \Phi$ is called an endpoint if it has a neighborhood relative to Φ whose closure is a simple arc with one endpoint at p. The set of singular points of an elementary curve is finite.

Let the elementary curve Φ be the union of the simple arcs $L_1, \cdots, L_s$. Subdividing these simple arcs, if necessary, into a finite number of smaller simple arcs, we may assume, without loss of generality, that each point contained in more than one of the arcs $L_1, \cdots, L_s$ is a common endpoint of all the arcs which contain it and that every two arcs have no more than one point in common. Let $a_1, \cdots, a_n$ be all the points which are endpoints of at least one of the arcs $L_1, \cdots, L_s$. The sequence $a_1, \cdots, a_n$ contains all the singular points of Φ, but not all the points $a_1, \cdots, a_n$ are necessarily singular. Let $b_1, \cdots, b_n$ be n points in the three-dimensional space, no four of which lie in a single plane; then, in particular, no three will lie on one line. If a_i and a_j are the endpoints of an arc L_h, let us join b_i and b_j by a segment L'_h. Two such segments L'_h and L'_k either have no points in common or have a common endpoint corresponding to the common endpoint of the arcs L_h and L_k.

It follows that the curve Φ is homeomorphic to a polygonal line, i.e., to a line which is the union of a finite number of segments having by pairs no common points except, perhaps, for common endpoints (Fig. 17).

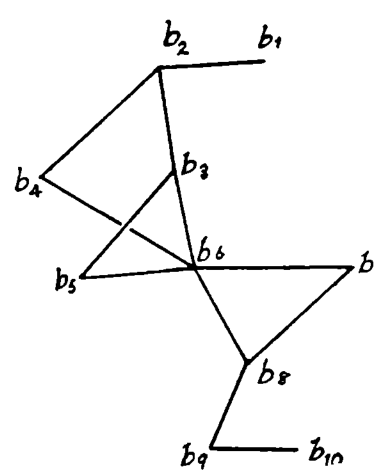


FIG. 17

Hence

1.12. *Every elementary curve is homeomorphic to a polygonal line in* R^3.

Therefore, the study of elementary curves is reduced to that of polygonal lines in R^3.

Let p be any regular point of a polygonal line Φ. It is either an interior point of exactly one segment or the common endpoint of exactly two segments, i.e., it is, in any case, an interior point of some simple arc of Φ consisting entirely of regular points. This simple arc can be extended in both directions until a singular point is reached or until a simple closed polygon is described. It follows easily from this that:

1.13. Every regular point p of an elementary curve Φ is an interior point of a uniquely determined curve $\Phi_p \subseteq \Phi$ which is:

either a simple closed curve, the component (I, 3.2) of p in Φ;

or a simple closed curve, containing exactly one singular point of Φ;

or a simple arc, both of whose endpoints are singular points of Φ.

The curve Φ_p is called the *regular component* of p.

A regular component of Φ contains no singular points if, and only if, it is both a simple closed curve and a component of Φ. Every other regular component contains one singular point if it is closed, and two if it is a simple arc.

Definition 1.14. The difference between the number of singular points of an elementary curve and the number of those of its regular components which contain singular points is called the *(Euler) characteristic* of the curve.

Since a topological mapping of an elementary curve preserves singular points and regular components, two homeomorphic curves have the same characteristic. It is also clear that the characteristic of a simple arc is 1, and that of a simple closed curve is 0. The characteristic of a lemniscate is -1.

1.15. Let an elementary curve Φ be decomposed into mutually nonintersecting open arcs and their endpoints; we shall call the resulting open arcs and their endpoints the one-dimensional and zero-dimensional elements (1-elements and 0-elements), respectively, of the decomposition of the curve Φ. This decomposition is referred to as a one-dimensional complex (1-complex) if no two open arcs have two endpoints in common.

Let us denote the number of 0-elements of the complex K by ρ_0 and the number of 1-elements by ρ_1. Then the number $\rho_0 - \rho_1$ *(the Euler characteristic of the complex* K) is equal to the Euler characteristic of the curve Φ.

Indeed, replacing two arcs (ab) and (bc), whose common vertex b is a regular point of the curve Φ, by the single arc (ac), decreases ρ_0 and ρ_1 by 1, so that $\rho_0 - \rho_1$ is unchanged. Repeating this process of "obliterating" the 0-elements of the complex a finite number of times, we finally arrive at the regular components, whence 1.15 follows without difficulty.

Obviously:

1.16. *The Euler characteristic of a curve Φ is equal to the sum of the Euler characteristics of its components.*

§1.2. The connectivity of a curve (the one-dimensional Betti number). Let us first make the following preliminary remark. Let $M_1, \cdots, M_s$ be a finite number of sets composed of perfectly arbitrary elements. The set $M = M_1 + \cdots + M_s$ consisting of those, and only those, elements each of which is contained in an odd number of the sets $M_1, \cdots, M_s$ is called the sum (mod 2) of these sets. If there are no such elements, the sum (mod 2) of the sets $M_1, \cdots, M_s$ is zero (i.e., the empty set):

$$M_1 + \cdots + M_s = 0.$$

Now let Φ be an elementary curve. Let us consider the following sets whose elements are regular components of Φ:

1. Sets consisting of a single element, viz., of a regular component which is a simple closed curve.
2. Sets of regular components whose union is a simple closed curve.
3. The empty set.

These sets of regular components are called *simple cycles*.

The sum (mod 2) of a number of simple cycles is called a *cycle*.

The set of all cycles of a curve Φ forms a group if the group operation is taken to be addition (mod 2). (In this connection, $-z = z$ for any cycle z, since $z + z = 0$.)

The cycles

$$z_1, \cdots, z_s \tag{1.211}$$

are *linearly dependent* if a subsequence of (1.211), say $z_{i_1}, \cdots, z_{i_n}$ (all the i_k distinct), has a sum (mod 2) equal to zero. [Instead of this we could say: if a linear combination of the cycles of (1.211), not all of whose coefficients $\equiv 0$ (mod 2), is equal to zero.]

In the contrary case, the cycles (1.211) are said to be *linearly independent*.

DEFINITION 1.21. The *connectivity* or *one-dimensional Betti number* $\pi^1(\Phi)$ of an elementary curve Φ is the maximum number n such that Φ has a system of n linearly independent cycles.

The number $\pi^1(\Phi)$ could also, obviously, be defined as the greatest number of linearly independent *simple* cycles in Φ.

REMARK. Let K be any decomposition of the elementary curve Φ into open arcs and their endpoints ("vertices").

Let us call every set of 1-elements of K which together with their vertices form a simple closed curve in Φ a simple cycle of the *decomposition* K; a

sum (mod 2) of a number of simple cycles we shall call, as before, a cycle. It is then clear that the cycles of K are in (1–1) correspondence with the cycles of the curve Φ, so that their respective groups are isomorphic; consequently, the connectivity of Φ is identical with the *connectivity of the complex* K, i.e., with the maximum number of linearly independent cycles of K.

Obviously:

1.22. The connectivity of a curve Φ is equal to the sum of the connectivities of the components of Φ.

1.23. If an elementary curve Φ_0 is a subset of an elementary curve Φ, then

$$\pi^1(\Phi_0) \leq \pi^1(\Phi).$$

We shall prove the following fundamental proposition, known as the Euler theorem (for curves).

1.24. *The Euler characteristic of an elementary curve is equal to the difference between the number of its components and its connectivity.*

Or, denoting by ρ_0 and ρ_1 the number of 0- and 1-elements of any decomposition K of the curve Φ, by π^0 the number of components of Φ, and by π^1 its one-dimensional Betti number, we have

$$\rho_0 - \rho_1 = \pi^0 - \pi^1. \tag{1.24}$$

For a connected curve, (1.24) becomes

$$\rho_0 - \rho_1 = 1 - \pi^1. \tag{1.241}$$

It suffices to prove (1.241). Then (1.24) follows for an arbitrary curve by 1.16 and 1.22.

Therefore let Φ be a connected curve. Without loss of generality, we may assume that Φ is a polygonal curve with the given decomposition K into straight line segments.

We shall reconstruct the given curve Φ in the following way: we shall begin the construction with an arbitrary segment and then add new segments one after another, so that at any time the curve already constructed is connected. This is clearly always possible. With each segment we shall also add those vertices of this segment which have not yet been added previously, if such vertices exist.

The addition of new segments may be of two kinds:

1. An addition of the first kind take place when, up to the addition, the figure already constructed contains only one of the two vertices of the segment to be added.

2. The addition is of the second kind if both of the vertices of the segment to be added have been added previously (Fig. 18).

It is easily seen that $\rho_0 - \rho_1$ and π^1 are invariant under an addition of the first kind.

An addition of the second kind obviously diminishes $\rho_0 - \rho_1$ by 1. We will show that an addition of this kind augments π^1 by 1, so that $1 - \pi^1$ decreases by 1; this will also prove (1.241) completely: at the beginning of the construction, i.e., for a curve consisting of a single closed segment, both sides of (1.241) were equal to 1, so that the formula was valid then; addition of a new segment did not disturb either side of (1.241), so that (1.241) remained true. Thus it remains to be shown that an addition of the second kind augments π^1 by 1.

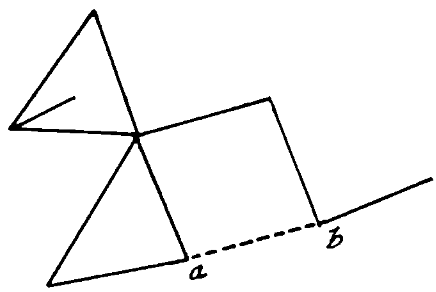


FIG. 18

Let the given addition consist in adding to the connected curve Φ_0 the open segment (ab), whose vertices a and b already belong to Φ_0. Since Φ_0 is connected, a and b can be joined by a broken line in Φ_0. This line, together with (ab), forms a cycle z_0 in Φ_0. Now let the connectivity of Φ_0 be $\pi^1_0 = r$, and let

$$z_1, \cdots, z_r$$

be a system of r linearly independent cycles of Φ_0. Since z_0 contains the segment (ab), which is not in any of the cycles $z_1, \cdots, z_r$, the cycles $z_0, z_1, \cdots, z_r$ are also linearly independent. It remains to be proved that the cycles

$$z, z_0, z_1, \cdots, z_r,$$

where z is an arbitrary cycle of the decomposition of $\Phi_0 \cup (ab)$, are linearly dependent. This is true by hypothesis, if the cycle z is in Φ_0. If z is not in Φ_0, it contains the segment (ab). Then the cycle $z + z_0$ is in Φ_0 and consequently is a sum of certain of the cycles $z_1, \cdots, z_r$, e.g., $z + z_0 = z_1 + \cdots + z_k$. Then $z + z_0 + z_1 + \cdots + z_k = 0$, which was to be proved.

REMARK. The number of additions of the second kind is independent of the choice of the decomposition K of the connected elementary curve Φ and of the order in which the segments of K are put together in the con-

struction of Φ, as long as the curve is connected at every stage of the construction. The number of such additions is equal to the connectivity of the curve.

Indeed, the assertion is obviously true for a curve consisting of one segment. Since the number π^1 increases by 1 for every addition of the second kind and does not change for an addition of the first kind, the assertion is true in general.

§2. Surfaces and their triangulations

§2.1. 2-complexes and polyhedra. By a triangle we shall mean the interior of a triangle; by a segment or *edge*, an open segment (a segment without its endpoints).

DEFINITION 2.11. A finite set K whose elements are triangles, segments, and individual points of R^n is called a *triangulation* if the following conditions are satisfied:

1. No two elements of K have points in common.

2. All the sides and vertices of any triangle of K and both vertices of every segment of K are elements of K.

In this chapter, "complex" will refer either to a triangulation or to any subset of a triangulation. The maximum dimension of the elements of a complex is called the *dimension* of the complex (in our case 0, 1, or 2).

The point set union in the given R^n of the points of all the elements of a complex K is called the *body of the complex* K and is denoted by $\| K \|$. *The body of a triangulation is called a polyhedron.* If the polyhedron Φ is the body of a triangulation K, $\Phi = \| K \|$, we shall say that K is a triangulation of the polyhedron Φ.

It is easily seen that polyhedra are bounded closed subsets of a given R^n, that is, polyhedra are compacta.

The elements of a complex K may be partially ordered as follows. Let $T \in K$. The vertices and sides of T, if T is a triangle, and the vertices of T, if T is a segment, shall precede T. If T' precedes T, we shall say that T follows T'.

A subcomplex K_0 of a complex K is said to be *closed* (*open*) if every element of K which precedes (follows) an element of K_0 is itself an element of K_0.

Obviously:

2.11. *A closed subcomplex of a triangulation is a triangulation.*

DEFINITION 2.12. A complex is said to be *connected* if it cannot be expressed as the union of two nonempty disjoint closed subcomplexes.

THEOREM 2.13. *In order that a triangulation be connected, it is necessary and sufficient that it be possible to join any two of its vertices by a broken line consisting of elements of the triangulation.*

Proof of necessity. Let a and b be two vertices of the triangulation K

which cannot be joined by a broken line in K. Let K_a be the set of all those elements of K of which at least one (and consequently every) vertex can be joined by a broken line to a; denote by K_b the remaining elements of K. It is easily seen that K_a and K_b are closed subcomplexes of K; they are nonvacuous, since $a \in K_a$, $b \in K_b$. This contradicts the hypothesis that K is connected.

Proof of sufficiency. If K is not connected and

$$K = K_1 \cap K_2, \qquad K_1 \cap K_2 = 0,$$

where K_1 and K_2 are closed subcomplexes of K, no vertex of K_1 can be joined to any vertex of K_2 by a broken line, which was to be proved.

If a polyhedron is connected, all its triangulations are connected (a partition of a triangulation into two nonempty disjoint closed subcomplexes corresponds to a partition of its body into two nonempty disjoint closed sets). Conversely, if a triangulation is connected, its body is also connected: it follows easily from 2.13 that every two points of $\| K \|$ can be joined by a broken line in $\| K \|$ (Fig. 19). Thus:

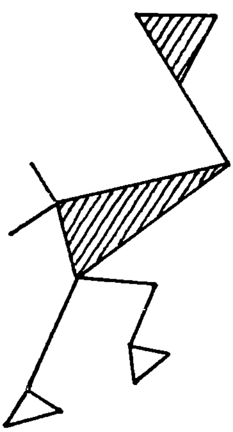


FIG. 19

2.14. *If at least one triangulation of a polyhedron is connected, the polyhedron is connected; if a polyhedron is connected, all its triangulations are connected.*

A two-dimensional complex (2-complex) is said to be *pure* (rein) if every 1- and 0-element of the complex precedes some 2-element. A pure 2-complex K is said to be *strongly connected* if every two triangles T_1 and T_s of K *can be connected by a chain of triangles*

$$T_1, \cdots, T_s$$

such that T_i and T_{i+1}, $i = 1, \cdots, s - 1$, have a common side in K.

Finally, let us introduce the following definition:

Let $T \in K$. The subcomplex consisting of T and of all elements of the complex K which follow T is called the *star* $O_K T$ *of* T *in* K.

Thus, if T is a triangle, $O_K T = T$; if T is a 1-element, $O_K T$ consists of T and of all the triangles of K which have T as a side; if T is a vertex, $O_K T$ consists of T and of all 1- and 2-elements of K having T as a vertex.

Examples of all these cases are given in Fig. 20.

Let us consider the special case that T is a vertex e of a triangulation K. Then the star $O_K e$ consists of the triangles of the form (ea_1a_2), of the segments of the form (ea), and of the vertex e. The sides (a_1a_2) of these tri-

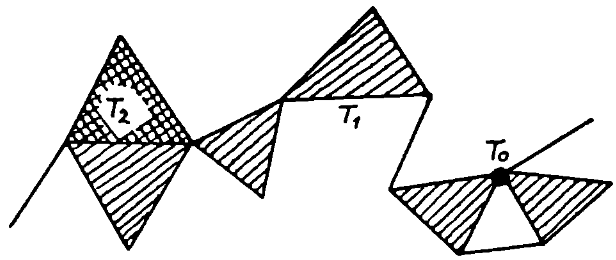


FIG. 20a

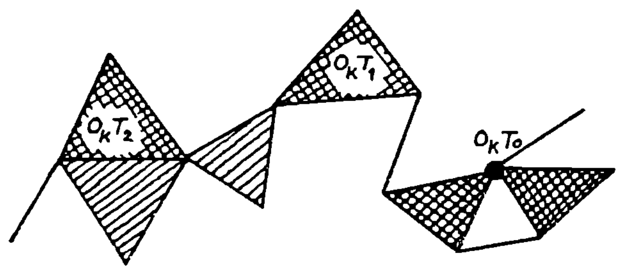


FIG. 20b

angles (opposite the vertex e) and the vertices a of the segments (ea) make up a complex $B_K e$ called the *outer boundary* of the star $O_K e$ (Fig. 20).

A star $O_K e$ is said to be *cyclic* if its outer boundary is a simple closed broken line (i.e., if its elements are disposed in cyclic order, like the elements of a circle split up into sectors). If the outer boundary of a star $O_K e$ is a simple broken line which is not closed, the elements of the star have a natural linear order and the star is said to be *semi-cyclic.*

For an example of a semi-cyclic star see Fig. 21.

§2.2. Closed surfaces. The following theorem will neither be proved nor used in this book: *Every closed* (*topological*) *2-manifold* (I, 5.3) *is homeomorphic to some polyhedron* (see Gawehn [a]). In order not to depend on this theorem, we shall at the very beginning restrict ourselves to those closed 2-manifolds which are homeomorphic to polyhedra and simply call them *closed surfaces.* The fact of the matter is that the class of closed surfaces defined in this way is synonymous with the class of all closed 2-manifolds.

If a triangulation K of a polyhedron Φ_0 homeomorphic to a given closed surface Φ, and a definite topological mapping C of the polyhedron Φ_0 onto the surface Φ are chosen, the triangulation K will be mapped onto a "curved triangulation" of the surface Φ, consisting of curvilinear triangles, their sides, and vertices. For example, if an octahedron or an icosahedron inscribed in a sphere is centrally projected onto the sphere, the sphere is

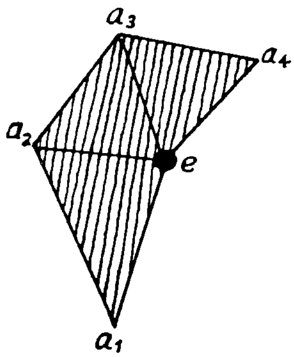


FIG. 21

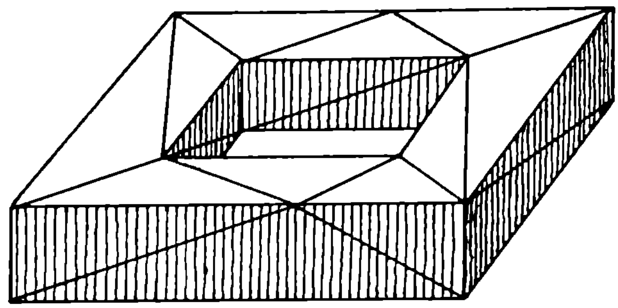


FIG. 22

decomposed, in the first case into 8, in the second case into 20, spherical triangles; these triangles, their sides, and vertices are the elements of the corresponding curved triangulation of the sphere. In exactly the same way, the triangulation of the polyhedron (Fig. 22) homeomorphic to a torus goes over into a curved triangulation of the torus under a topological mapping.

REMARK. Since every closed surface is homeomorphic to a polyhedron, the topological study of surfaces can be confined to those surfaces which are themselves polyhedra. We shall, therefore, without further reservations, assume in the sequel that all surfaces considered are polyhedra. This will, in particular, enable us to avoid completely the use of curved triangulations.

We shall accept without proof the following theorem, which will be proved in the sequel (V, 3.140).

2.20. *Let A and B be two subsets of a topological space R, each of which is homeomorphic to the plane. Then if $A \subseteq B$, A is open in B.*

We can now prove the following fundamental theorem:

2.21. *In order that a triangulation K be a triangulation of a closed surface, it is necessary and sufficient that K be connected and that the star of every vertex of K be cyclic.*

Proof of necessity. Let K be a triangulation of the closed surface $\Phi = \| K \|$. By 2.14, the connectedness of K follows from that of Φ. Before proving that every star $O_K e$, where e is a vertex of K, is cyclic, we shall prove that every segment T^1 of K is a side of more than one triangle. This assertion obviously follows from:

2.210. Let T be a plane triangle, let $\overline{T}$ be the closure of T in R^2, and let $p \in \overline{T} \setminus T$. Then there is no set Γ homeomorphic to the plane such that

$$p \in \Gamma \subseteq \overline{T}.$$

Indeed, suppose that such a set Γ exists. Then Γ and R^2 satisfy the conditions on the sets A and B, respectively, of Theorem 2.20. Hence Γ is

open in R^2 and p is an interior point of Γ. Thus, a fortiori, p is an interior point of $\overline{T}$ relative to R^2, which is obviously not so. This proves 2.210.

The fact that every 1-element of K is a side of at least two triangles immediately implies that there is a cyclicly ordered sequence of triangles of $O_K e$ such that each pair of neighboring elements of the sequence has a common side of the form (ee_i). These triangles, their common sides (ee_i), and e make up a subcomplex O' of $O_K e$ and it suffices to prove that $O' = O_K e$. To this end, let Γ be a neighborhood of e relative to Φ homeomorphic to R^2.

Let us put

$$\epsilon = \rho(e, \Phi \setminus \Gamma) > 0$$

and denote by d the maximum of the diameters of the elements of the complex O'. Let $0 < \mu < \epsilon/d$. A similitude with center e and coefficient μ takes the complex O' into a complex O'' such that $e \in \| O'' \| \subseteq \Gamma$ (see

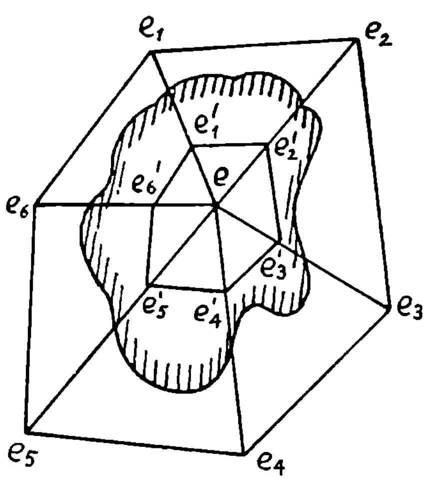


FIG. 23

Fig. 23 which shows the complex O', the intersection of its elements with Γ, and the complex O''). Both $\| O' \|$ and $\| O'' \|$, as is easily seen, are homeomorphic to the interior of a circle and, hence, to the plane. Therefore, the sets $\| O'' \|$ and Γ satisfy the conditions on the sets A and B of Theorem 2.20. $\| O'' \|$ must be open in Γ. But if there were an element T of $O_K e$ not in O', this element would not intersect $\| O' \|$ and, all the more, $\| O'' \|$; on the other hand, e is a limit point of T and hence cannot be an interior point of $\| O'' \|$ relative to Γ. This contradiction proves that $O' = O_K e$.

Proof of sufficiency. Since K is connected, $\| K \|$ is also connected; we shall prove that the fact that every star $O_K e$ is cyclic implies that every point $p \in \Phi$ is contained in an open subset of the polyhedron Φ homeomorphic to the plane or, what comes to the same, to the interior of a circle.

First, $O_K e$ is, by definition, an open subcomplex of K and the comple-

mentary subcomplex $K \setminus O_K e$ is a closed subcomplex of K. Hence $\| K \setminus O_K e \|$, being a polyhedron, is closed in $\Phi = \| K \|$, while $\| O_K e \|$ is open in Φ. The set $\| O_K e \|$ is obviously homeomorphic to the interior of a circle. Now let p be any point of Φ and T the unique element of K which contains p. Let e be any vertex of T. Then $p \in \| O_K e \|$. This completes the proof.

2.22. *Every 1-element T^1 of a triangulation K of a closed surface Φ is a side of exactly two triangles of K.*

Indeed, if e is a vertex of the segment T^1, all the triangles of K adjoining T^1 are elements of the cyclic star $O_K e$; since $O_K e$ contains precisely two triangles with side T^1, 2.22 is proved.

§2.3. Surfaces with boundary. In this chapter, we shall understand by a surface a compactum Φ homeomorphic to a polyhedron and with the following property. The points of Φ can be split into two classes: *interior points*, having neighborhoods homeomorphic to the plane, and *boundary points*; in this connection, a point $p \in \Phi$ is called a *boundary point* if it has a neighborhood which can be mapped topologically onto the union of a triangle T^2 and one of its sides T^1 in such a way that the mapping takes p into a point of the open segment T^1.

The set of all boundary points of a surface is called the *boundary of the surface*. If it is nonempty, the surface is called a *surface with boundary*; in the contrary case, the surface is closed in the sense of the preceding article. In agreement with the notation of the preceding article, we will always assume in the sequel that the surfaces considered are themselves polyhedra.

2.210 implies

2.31. *The division of the points of a surface into interior and boundary points is, in fact, a decomposition into two disjoint classes: the boundary points of a surface are characterized as those which do not have neighborhoods homeomorphic to the plane; a topological mapping of one surface onto another preserves interior and boundary points.*

It follows from the very definition of boundary point that every boundary point of a surface is an interior point, relative to the boundary of the surface, of some simple arc consisting entirely of boundary points. On the other hand, for an arbitrary choice of a triangulation K of the surface Φ all the boundary points of Φ lie either on the 1- or on the 0-elements of K; it follows from these two observations that the boundary of a surface is an elementary curve without singular points and, therefore, is the union of simple closed curves, no two of which have points in common. These simple closed curves, which are the components of the boundary of the surface, are called the *contours* of the surface.

Let p be a boundary point of the surface Φ and K an arbitrary triangulation of Φ. The point p is either contained in a segment T^1 of K, or is a vertex of K. In the first case, by 2.210, T^1 is a side of exactly one triangle of K and all the points of T^1 are boundary points. In the second case p, as a vertex of K, is the common endpoint of exactly two segments (pe_1) and (pe_s) of K on the boundary of Φ, each of which is a side of one triangle of K. Since all the remaining elements of the star $O_K p$, except for the point p and the segments (pe_1) and (pe_s), consist of interior points of the surface, every segment of the form (pe_i), $1 \neq i \neq s$, is a side of two triangles of K. It follows easily that every star $O_K p$ is semi-cyclic. Conversely, if the star of a vertex e of K is semi-cyclic, e is a boundary point. Therefore,

2.32. *Let K be any triangulation of a surface Φ. In order that a point p be a boundary point it is necessary and sufficient that it satisfy one of the following two conditions*:

a) *p is contained in a segment of K which is a side of only one triangle of K*;

b) *p is a vertex of a semi-cyclic star of K.*

Those elements (edges and vertices) of K which are on the boundary of Φ are called boundary elements of K. The remaining elements are called interior elements. The boundary of Φ is the union of the boundary elements of any triangulation of Φ.

2.33. *A triangulation of a surface is a strongly connected complex.*

Indeed, let T^2_1 and T^2_s be two triangles of K and suppose it impossible to connect them by a chain of triangles

$$T^2_1, \cdots, T^2_s \tag{2.33}$$

such that T^2_i and T^2_{i+1} have a common side. Let K_1 be the subcomplex of K consisting of the triangles which can be connected to T^2_1 by such chains and of the sides and vertices of these triangles; let K_2 be the subcomplex of K consisting of all the remaining triangles, edges, and vertices of K. K_1 and K_2 are disjoint, by definition. On the other hand, since cyclic and semi-cyclic stars are strongly connected, the star of every vertex of K_1 (relative to K) is wholly contained in K_1 and the star of every vertex of K_2 (relative to K) is wholly contained in K_2. Therefore, K_1 and K_2 are complementary open, and hence closed, nonempty subcomplexes of K. This contradicts the fact that K is connected.

§2.4. Subdivisions of triangulations. It is sometimes necessary to pass from a given triangulation of a surface to a finer triangulation by subdividing the original one. We shall consider only the following types of subdivisions:

1. *An elementary subdivision* of the triangulation of a surface relative to a given 1-element T^1 of the triangulation consists in dividing T^1 into two

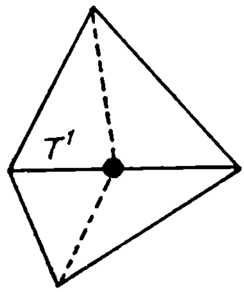


FIG. 24

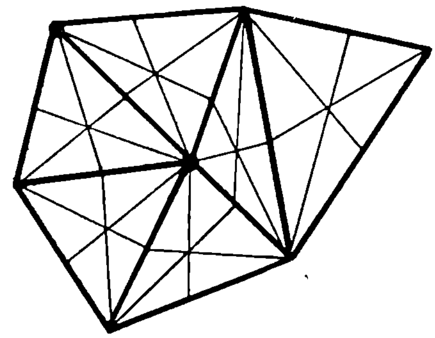


FIG. 25

segments and then dividing each of the triangles adjoining T^1 into two triangles, as in Fig. 24; all the remaining elements of the triangulation remain unchanged.

2. The result of performing an elementary subdivision relative to all the 1-elements of a triangulation is a *barycentric subdivision* of the triangulation, whose structure is clear from Fig. 25 (a barycentric subdivision of a triangle consists of the six triangles into which the given triangle is divided by its three medians).

2.41. A subdivision of a given triangulation, obtained as the result of a finite number of elementary subdivisions, is said to be a *regular subdivision.*

In this book it will be necessary to consider only regular subdivisions. It is easy to show that a barycentric subdivision is regular.

§2.5. Skeleton complexes. The set consisting of the three vertices of a triangle is called the *skeleton* [in German: Gerüst; sometimes called an abstract simplex (see IV, Def. 1.41)] of the triangle; the set consisting of the two vertices of a segment is called the skeleton of the segment; a vertex is its own skeleton. The elements of a triangulation K correspond (1-1) to their skeletons and one element of K precedes another if, and only if, the skeleton of the first element is a proper subset of the skeleton of the second element.

Hence the study of a triangulation can be successfully replaced by the study of the set of all skeletons of the triangulation, that is, the *skeleton complex of the triangulation.*

In this connection, we shall make the following definitions.

DEFINITION 2.51. Let E be any set. The elements of E will be called vertices. Let us assume that certain subsets of E, called *skeletons,* consisting of one, two, or three vertices, have been singled out. The number of vertices in a skeleton less 1 is said to be its *dimension.* In this chapter a set of skeletons will be called a *skeleton complex* if it satisfies the following conditions:

a) Every nonempty subset of a skeleton is a skeleton.

b) Every set consisting of one vertex is a skeleton.

The maximum dimension (0, 1, or 2) of the elements of a skeleton complex is called the dimension of the complex.

REMARK. Higher dimensional skeleton complexes will be introduced in Chapter IV.

DEFINITION 2.52. Two skeleton complexes K and K' are said to be *isomorphic* if there is a (1–1) mapping of the set of vertices of one complex onto the set of vertices of the other complex which preserves skeletons.

A skeleton complex K and a triangulation K' are said to be isomorphic if K and the skeleton complex of K' are isomorphic. Two triangulations are isomorphic if their skeleton complexes are isomorphic.

The following important theorem is a special case of a theorem proved in IV, 1.9:

2.53. *Every two-dimensional skeleton complex is isomorphic to some triangulation in* R^5.

§3. Cuts and identifications

§3.1. Identification of elements in skeleton complexes. Let K be a skeleton complex. Let us divide the set E of all vertices of K into mutually disjoint subsets or classes $E'_1, \cdots, E'_s$; and let us assign a new "vertex" e'_i to each class E'_i, with the agreement that a pair or triple of vertices e'_i shall form a skeleton if, and only if, the corresponding classes E'_i contain vertices e_i forming a skeleton of K.

The skeletons defined in this way form a skeleton complex K'; *the complex K' is said to be obtained from the complex K by identification of certain vertices of K*; each vertex e'_i of K' is obtained by identifying all the vertices of K contained in the class E'_i.

Now let K be any triangulation and let K_1 be the skeleton complex obtained by identifying certain vertices of the skeleton complex of the triangulation K. Let K' be any triangulation isomorphic to K_1 (such triangulations exist by virtue of Theorem 2.53).

The *triangulation K'* is said *to arise from the triangulation K through identification of certain elements of K.*

An identification of the elements of a skeleton complex (or triangulation) K obviously induces a mapping of the set of all vertices of the complex K onto the set of all vertices of the complex K': if $e_n \in E'_i$, the vertex e'_i is assigned to the vertex e_n. *This vertex mapping maps every skeleton of the complex K onto a skeleton of the complex K'* and maps all of K onto K'. A mapping of this kind is called a *simplicial mapping* (for details see IV, 1.6) of the triangulation K onto the triangulation K'. *In this chapter we shall consider only identifications which satisfy the following conditions*:

1. No two vertices which form a skeleton of K are mapped into a single vertex of K'.

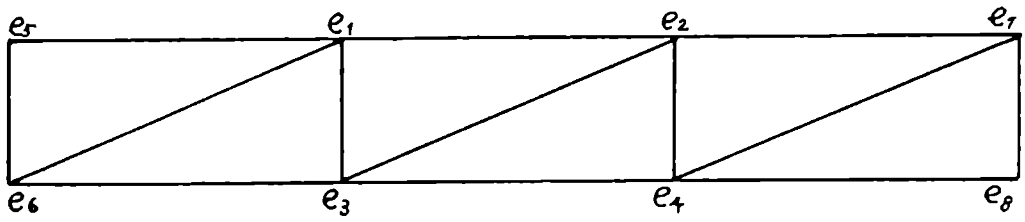


FIG. 26

2. Every triangle of K' is the image of exactly one triangle of K.

3. Every segment of K' is the image of at most two segments of K.

EXAMPLES. 1. Consider the triangulation of the rectangle shown in Fig. 26. Let us divide the set of all eight vertices of this triangulation into the following classes: each of the classes E'_i, $i = 1, 2, 3, 4$, consists of the one vertex e_i; E'_5 consists of the vertices e_5 and e_7, and E'_6 of e_6 and e_8. This identification converts the triangulation K of the rectangle into a triangulation K' of the lateral surface of a triangular prism (Fig. 27). Both segments (e_5e_6) and (e_7e_8) of K correspond to the same segment $(e'_5e'_6)$ of K'.

Since the lateral surface of the prism is homeomorphic to the lateral surface of a cylinder and also to a plane circular ring, we may say that the identification just described converts the triangulation K of the rectangle into a triangulation K' of a cylinder or a plane ring.

2. The following identification differs from the preceding one only in that now the class E'_5 consists of the vertices e_5 and e_8, and the class E'_6 of e_6 and e_7. This identification converts the triangulation K of the rectangle into a triangulation K' of a surface known as a *Möbius band* (Fig. 28).

REMARK. We shall briefly refer to case 1 as an identification of the directed sides $(e_5e_6)^{\rightarrow}$ and $(e_7e_8)^{\rightarrow}$ of the rectangle and to case 2 as an identification of the directed sides $(e_5e_6)^{\rightarrow}$ and $(e_8e_7)^{\rightarrow}$. We shall use analogous terminology in other similar cases. A directed segment will be denoted by parentheses followed by an arrow.

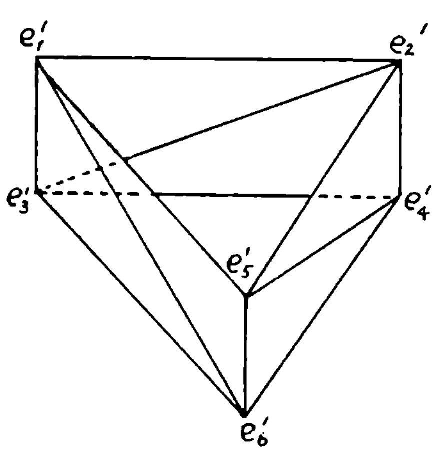


FIG. 27

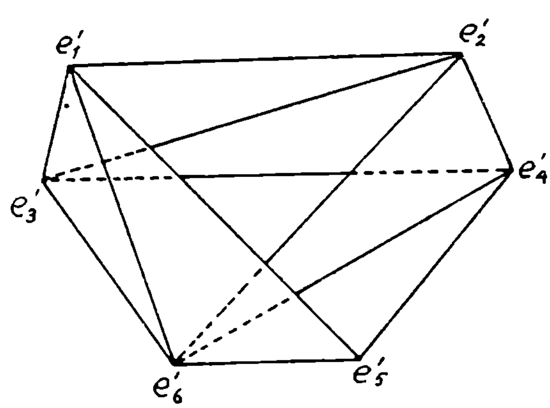


FIG. 28

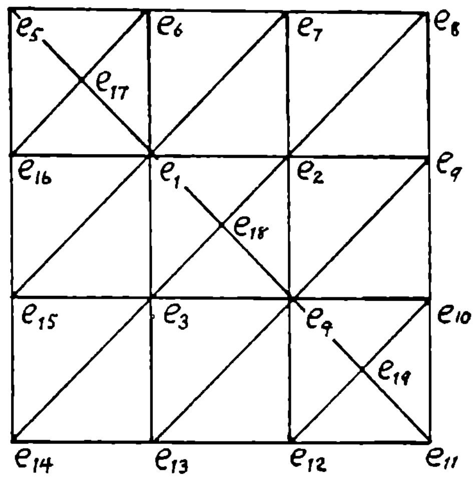


FIG. 29

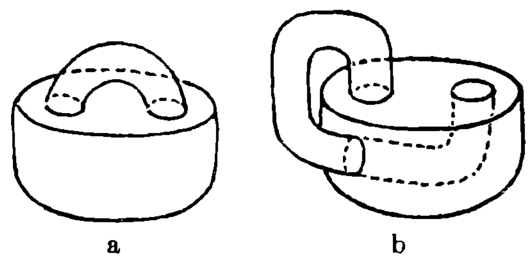


FIG. 30

3. Let us consider the triangulation of a rectangle shown in Fig. 29. Let us divide its set of 19 vertices into classes E'_i as follows: set $E'_i = e_i$, $i = 1, 2, 3, 4, 17, 18, 19$, and let

$$E'_5 = \{e_5, e_8, e_{11}, e_{14}\}, \qquad E'_6 = \{e_6, e_{13}\}, \qquad E'_7 = \{e_7, e_{12}\},$$
$$E'_8 = \{e_9, e_{16}\}, \qquad E'_9 = \{e_{10}, e_{15}\}.$$

This identification of the directed side $(e_5e_{14})^{\rightarrow}$ with the directed side $(e_8e_{11})^{\rightarrow}$ and of $(e_5e_8)^{\rightarrow}$ with $(e_{14}e_{11})^{\rightarrow}$ converts the original triangulation of the rectangle into a triangulation of the torus (Fig. 30a).

4. The following identification differs from the preceding only in that now

$$E'_6 = \{e_6, e_{12}\}, \qquad E'_7 = \{e_7, e_{13}\},$$

i.e., $(e_5e_{14})^{\rightarrow}$, as before, is identified with $(e_8e_{11})^{\rightarrow}$, but $(e_5e_8)^{\rightarrow}$ is now identified with $(e_{11}e_{14})^{\rightarrow}$. This identification yields a triangulation of a surface known as a *Klein bottle* (Fig. 30b), which cannot be imbedded in three-dimensional space.

5. Let us now set (Fig. 29)

$$E'_i = \{e_i\}, \qquad i = 1, 2, 3, 4, 17, 18, 19; \qquad E'_5 = \{e_5, e_{11}\};$$
$$E'_6 = \{e_6, e_{12}\}; \qquad E'_7 = \{e_7, e_{13}\}; \qquad E'_8 = \{e_8, e_{14}\};$$
$$E'_9 = \{e_9, e_{15}\}; \qquad E'_{10} = \{e_{10}, e_{16}\}$$

[so that the side $(e_5e_6)^{\rightarrow}$ is identified with the side $(e_{11}e_{12})^{\rightarrow}$, $(e_6e_7)^{\rightarrow}$ with $(e_{12}e_{13})^{\rightarrow}$, $\cdots$, $(e_{10}e_{11})^{\rightarrow}$ with $(e_{16}e_5)^{\rightarrow}$]. This identification converts the given triangulation of the rectangle into a triangulation of the *projective plane*. To show this, let us map the rectangle topologically onto the upper hemisphere of the sphere $x^2 + y^2 + z^2 = 1$ in such a way that the images of the

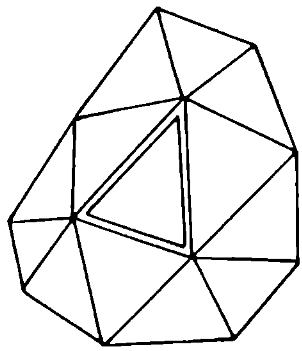


FIG. 31a

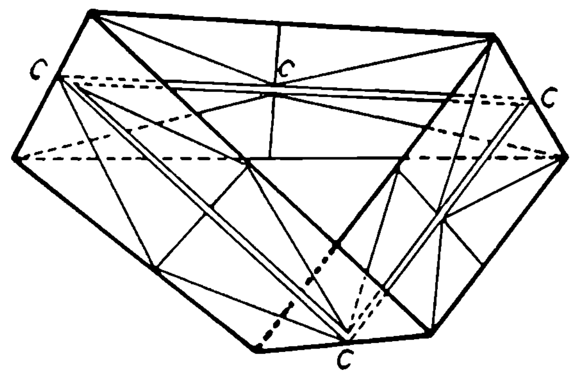


FIG. 31b

points e_5, e_6, $\cdots$, e_{16} divide the equator into 12 equal arcs. Then the identification can be interpreted as the identification of each pair of diametrically opposite points of the equator. But this (I, 5.2, 8) converts the hemisphere into a topological space homeomorphic to the projective plane.

REMARK. The stars of all the vertices of the above triangulation of the projective plane are obviously cyclic stars. It follows that the projective plane is a closed surface.

§3.2. Cut lines and semi-stars of the vertices of cut lines. (Only cut lines are discussed in this article; the cut operation is not introduced until 3.3. Therefore, the doubling of the segments in the figures of 3.2 should be ignored; this doubling will have no significance until 3.3.) Let $\Phi = \| K \|$ be a surface with a definite triangulation K. In this section, a *cut line* will mean a simple broken line Λ consisting of elements of the triangulation K and satisfying one of the following conditions:

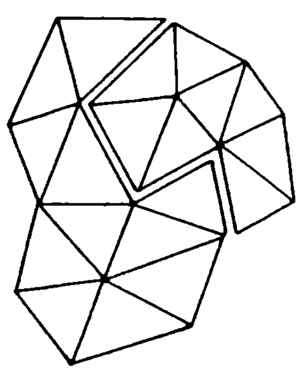


FIG. 32

1. Λ is a closed broken line, all of whose elements are interior elements of the triangulation K (i.e., elements not on the boundary of the surface Φ). This is the case of *interior closed cuts* (Fig. 31).

2. Λ is a simple nonclosed broken line, all of whose elements are interior elements of K except both endpoints of the broken line, which are on the boundary of the surface (*cross cut*, Fig. 32).

3. Λ is a simple broken line which is not closed and contains at least two links. One of the endpoints of Λ *may* be on the boundary of the surface; but none of the other elements of Λ are on the boundary (*open cut*, Fig. 33).

Let us consider any star $O_K e$ whose center e is one of the vertices of a cut

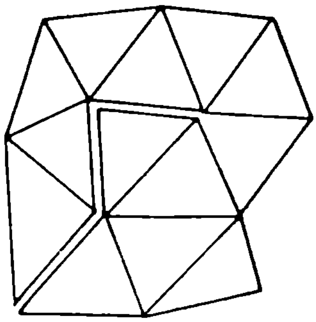


FIG. 33

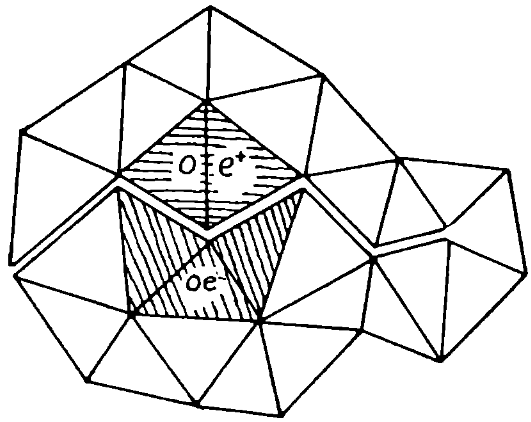


FIG. 34

line Λ; if Λ is an open cut line, let us assume in addition that the vertex e is different from the interior endpoint (or endpoints) of the open cut (Fig. 34). Under these assumptions, the set of all elements of the star $O_K e$ not contained in Λ is the union of two disjoint connected complexes, the two *semi-stars* of e; if one of these semi-stars is denoted by Oe^+, the other will be denoted by Oe^- (Fig. 34).

It is easily seen that:

3.21. If e and e' are two consecutive vertices of Λ, each of the two semi-stars of the vertex e meets precisely one of the semi-stars of the vertex e' (the intersection in each case consists of a triangle having (ee') as a side).

Let us note further that:

If Λ is an open cut line, e an interior endpoint of Λ, and e' the vertex of Λ adjacent to e, then $O_K e$ intersects each of the semi-stars of the vertex e' and each intersection is a triangle having (ee') as a side (Fig. 33).

It follows that:

3.220. If $[e_1 \cdots e_s]$ is an open cut line, one of whose endpoints, say e_1, is on the boundary of the surface, the semi-stars of the vertices $e_1, \cdots, e_{s-1}$ and the star of the vertex e_s form a chain

$$Oe^+_1, \cdots, Oe^+_{s-1}, O_K e_s, Oe^-_{s-1}, \cdots, Oe^-_1 .$$

[A chain (or closed chain) of subcomplexes of a given complex, here and throughout this chapter, is to be taken in the sense of I, Def. 3.14.]

If both endpoints e_1 and e_s of the open cut line are interior points, the stars $O_K e_1$ and $O_K e_s$ together with the semi-stars of the remaining vertices form one closed chain

$$O_K e_1, Oe^+_2, \cdots, Oe^+_{s-1}, O_K e_s, Oe^-_{s-1}, \cdots, Oe^-_2, O_K e_1 .$$

3.220 in turn implies

3.22. The complement in a triangulation K of an open cut line is a strongly connected open subcomplex of K.

In fact, every chain of triangles connecting two triangles T_1 and T_ν of K can always be replaced by a chain of triangles connecting T_1 and T_ν along the circuit of the open cut line (Fig. 35).

REMARK. A chain of triangles of a complex K connecting two triangles T_1, $T_\nu \in K$ will always mean a sequence of triangles $T_1, \cdots, T_\nu$ of K such that T_i and T_{i+1} $(1 \leq i \leq \nu - 1)$ have a common side in K.

Now let $\Lambda = [e_1 \cdots e_s]$ be a cross cut, and let $Oe^+{}_1$ be one of the two semi-stars of the vertex e_1. It intersects a completely determined semi-star of

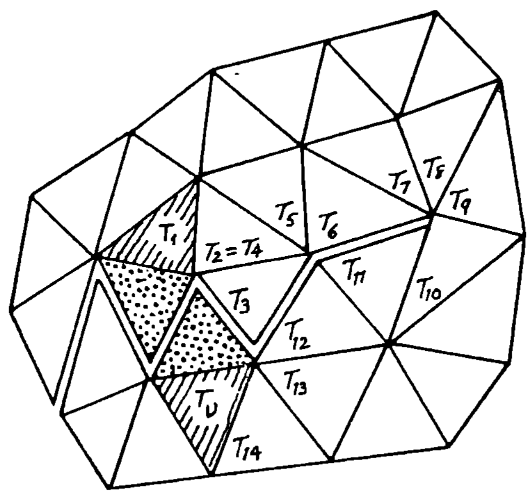


FIG. 35

the vertex e_2, which we shall denote by $Oe^+{}_2$; $Oe^+{}_2$ in turn intersects a definite semi-star of e_3, $Oe^+{}_3$, etc. We obtain, in this way, a chain of semi-stars

$$Oe^+{}_1, \cdots, Oe^+{}_s. \tag{3.23$_+$}$$

Similarly, starting with the semi-star $Oe^-{}_1$, we obtain a chain of semi-stars

$$Oe^-{}_1, \cdots, Oe^-{}_s. \tag{3.23$_-$}$$

Hence

3.230. *The semi-stars of the vertices of a cross cut form two chains,* (3.23$_+$) *and* (3.23$_-$).

From 3.230 we shall derive

3.23. *The complement in a triangulation K of a cross cut is an open subcomplex of K which is either itself strongly connected, or is the union of two disjoint strongly connected complexes. If the endpoints of the cross cut Λ lie on two different contours, $K \setminus \Lambda$ is always a strongly connected complex.*

To prove the first assertion, it suffices to show that an arbitrary triangle $T_1 \in K$ can be connected with a triangle of one of the semi-stars of (3.23$_+$) or (3.23$_-$) by a chain of triangles, consecutive members of which have a common side not contained in Λ. To this end let T_h be a triangle of, say,

one of the semi-stars of (3.23_+). Let us connect T_1 and T_h by a chain of triangles of K. If every pair of consecutive triangles of this chain has a common side not contained in Λ, the proof is complete; in the contrary case, let T_i, T_{i+1} be the first pair of triangles of the chain with a common side contained in Λ. Then T_i is contained in one of the semi-stars of (3.23_+) or (3.23_-) and $T_1, \cdots, T_i$ is the desired chain.

Let us now pass to the second assertion of Theorem 3.23, i.e., if all the elements of the cross cut Λ, whose endpoints belong to different contours π_1 and π_2 of the surface $\| K \|$, are deleted from K, the remainder is a strongly connected complex.

This assertion is readily proved by the methods used above; it is merely necessary to make the following very obvious remark. Under the given conditions, all the semi-stars of the vertices of the cross cut $\Lambda = [e_1 \cdots e_s]$ together with the stars $O_K e$ of all the vertices, different from e_1 and e_s, of the contours π_1 and π_2 form a single closed chain (Fig. 40).

Finally, let

$$\Lambda = \langle e_1 \cdots e_s e_1 \rangle$$

be an interior closed cut line. Let Oe^+_1 denote one of the two semi-stars of e_1, Oe^+_2 the unique semi-star of e_2 which intersects Oe^+_1, Oe^+_3 the unique semi-star of e_3 which meets Oe^+_2, etc., until e_s is reached. This yields a chain of semi-stars

$$Oe^+_1, \cdots, Oe^+_s. \tag{3.241}$$

Since e_s and e_1 are adjoining vertices, Oe^+_s meets a definite unique semi-star of e_1. Then two cases are possible:

1. Oe^+_s meets Oe^+_1.
2. Oe^+_s meets Oe^-_1, where Oe^-_1 is different from Oe^+_1.

In case 1 the chain (3.241) is closed; in this case, beginning with the semi-star Oe^-_1 and reasoning as before, we obtain a second closed chain of semi-stars

$$Oe^-_1, \cdots, Oe^-_s$$

and call the broken line Λ a *closed two-sided cut line* (Fig. 31a or Fig. 41).

In case 2 we obtain a closed chain

$$Oe^+_1, \cdots, Oe^+_s, Oe^-_1;$$

Oe^-_1 meets one definite semi-star of e_2, which differs from Oe^+_2 (since Oe^+_2 meets Oe^+_1), and hence is Oe^-_2; in exactly the same way, Oe^-_2 meets Oe^-_3, but not Oe^+_3. Continuing in this way, the previous chain of semi-stars is extended to the chain

$$Oe^+_1, Oe^+_2, \cdots, Oe^+_s, Oe^-_1, \cdots, Oe^-_s.$$

The semi-star Oe^-_s intersects one of the two semi-stars of e_1 and since it does not intersect Oe^-_1 (Oe^-_1 meets Oe^+_s), it intersects Oe^+_1. Hence, in case 2, every semi-star occurs in the closed chain

$$Oe^+_1, \cdots, Oe^+_s, Oe^-_1, \cdots, Oe^-_s, Oe^+_1. \tag{3.242}$$

In case 2, the broken line is called a *closed one-sided cut line* (see Figs. 31b and 42a; the reference is to the "center line" of the Möbius band which can be seen in Fig. 31b; this is the line cccccc, drawn as a double line). A slight change in the proof of 3.22 and 3.23 yields

3.24. The complement of a closed one-sided cut line in a triangulation K is a strongly connected open subcomplex of K.

3.25. The complement in a triangulation K of a closed two-sided cut line is either a strongly connected open subcomplex of K or the union of two disjoint strongly connected open subcomplexes of K.

In the triangulation of the projective plane (segments and vertices denoted by the same symbols are to be identified) shown in Fig. 36 the closed broken line $\langle e_1e_2e_3e_4e_5e_6e_1\rangle$ is a one-sided cut line. The two semi-stars of e_1 are distinguished by the signs $+$ and $-$, respectively.

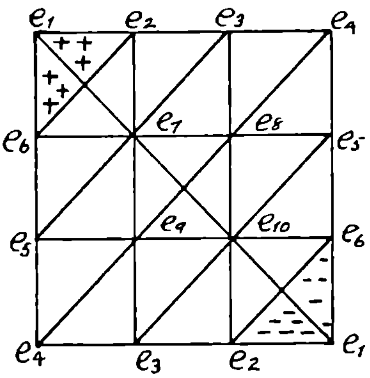


FIG. 36

§3.3. The cut operation. Having disposed of the preliminary definitions, propositions, and examples, we may now go on from cut lines to the *cut operation* itself. This operation consists of a passage from a given triangulation K to a new triangulation K', which depends on the cut line Λ and is defined only up to an isomorphism. It obviously suffices to define the skeleton complex of the triangulation K'.

We shall first define the vertices of K'. If Λ is an open cut, the vertices of K' are: 1) all the vertices of K not contained in Λ and the interior endpoint (or endpoints) of Λ; and 2) "new vertices", e'_i, corresponding (1-1) to all the semi-stars of the vertices of Λ. If Λ is a cross cut or a closed interior cut, the vertices of K' are: 1) all the vertices of K not in Λ; and 2) new vertices e'_i, corresponding (1-1) to the semi-stars of the vertices of Λ.

This definition of the vertices of K' implies that every new vertex of K' corresponds to a unique vertex e_i of Λ, i.e., to that vertex e_i, one of whose semi-stars corresponds to e'_i; it is clear, on the other hand, that each vertex e_i of the broken line (with the exception of one or both endpoints, if Λ is an open cut) is placed in correspondence with precisely two new vertices e'_i (the number of semi-stars of e_i). These two new vertices e'_i which

correspond to the same vertex e_i of Λ will, as the need arises, be denoted by $(e'_i)_+$ and $(e'_i)_-$, or simply by e'_{i+} and e'_{i-}.

The elements of the skeleton complex K' are defined as follows: 1) every skeleton of K, all of whose vertices are in K', is a skeleton of K'; 2) skeletons of the form $\{e'_i, e_j\}$ and $\{e'_i, e_j, e_k\}$ are in K' if, and only if, the semi-star corresponding to the vertex e'_i contains $\{e_i, e_j\}$ or $\{e_i, e_j, e_k\}$, respectively; 3) if Λ is an open cut, skeletons of the form $\{e'_k, e_l\}$ are in K' where e_l is an interior endpoint of the open cut line and e_k is the vertex of Λ adjacent to it; 4) skeletons of the form $\{e'_i, e'_j\}$, $\{e'_i, e'_j, e_k\}$, and $\{e'_i, e'_j, e'_k\}$ are in K' if, and only if, the semi-stars corresponding to the vertices e'_i, e'_j, and (in the last case) e'_k intersect and (in the second case) this intersection contains $\{e_i, e_j, e_k\}$.

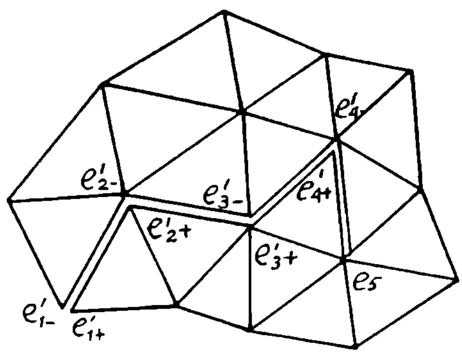


FIG. 37

It is immediately verified that the stars of all the vertices of K' are cyclic or semi-cyclic. Hence K' is a triangulation of a surface or of several (in virtue of 3.22–3.25, no more than two) disjoint surfaces.

It follows readily from the definition of the skeletons of the triangulation K' and from the propositions proved in this article that:

3.31. An open cut one of whose endpoints e_1 lies on the boundary of the surface transforms the broken line $\Lambda = [e_1 \cdots e_{s+1}]$ into a simple nonclosed broken line

$$[e'_{1+}e'_{2+} \cdots e'_{s+}e_{s+1}e'_{s-}e'_{(s-1)-} \cdots e'_{2-}e'_{1-}]$$

which is part of one of the contours of the surface K' (Fig. 37).

K and K' have the same number of contours.

An open cut $\Lambda = [e_1 \cdots e_{s+1}]$ both of whose endpoints are interior points transforms the nonclosed broken line Λ into a closed broken line

$$\langle e_1e'_{2+} \cdots e'_{s+}e_{s+1}e'_{s-} \cdots e'_{2-}e_1\rangle$$

which appears as a new contour of K'. The number of contours of K' is one more than that of K (Fig. 38).

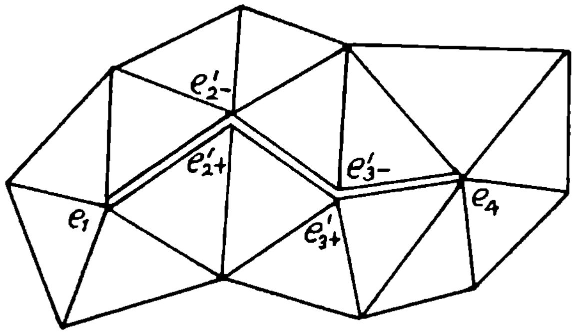


FIG. 38

3.32. A cross cut $\Lambda = [e_1 \cdots e_s]$ replaces the nonclosed broken line Λ by two nonclosed broken lines

$$[e'_{1+} \cdots e'_{s+}] \quad \text{and} \quad [e'_{1-} \cdots e'_{s-}].$$

If both endpoints of the broken line are on the same contour of the surface K, K' may have one more contour than K (Fig. 39). Here the contour $\langle e_1 a_1 \cdots a_\mu e_s b_1 \cdots b_\nu e_1 \rangle$ is replaced by the two contours

$$\langle e'_{1-} a_1 \cdots a_\mu e'_{s-} e'_{(s-1)-} \cdots e_{2-} e'_{1-} \rangle$$

and

$$\langle e'_{1+} e'_{2+} \cdots e'_{(s-1)+} e'_{s+} b_1 \cdots b_\nu e'_{1+} \rangle.$$

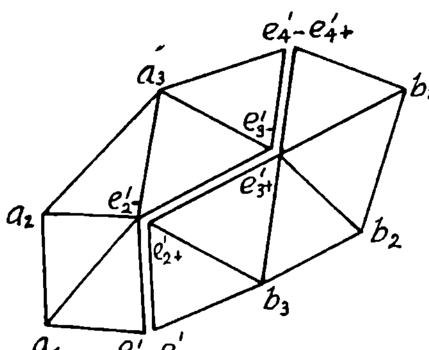


FIG. 39

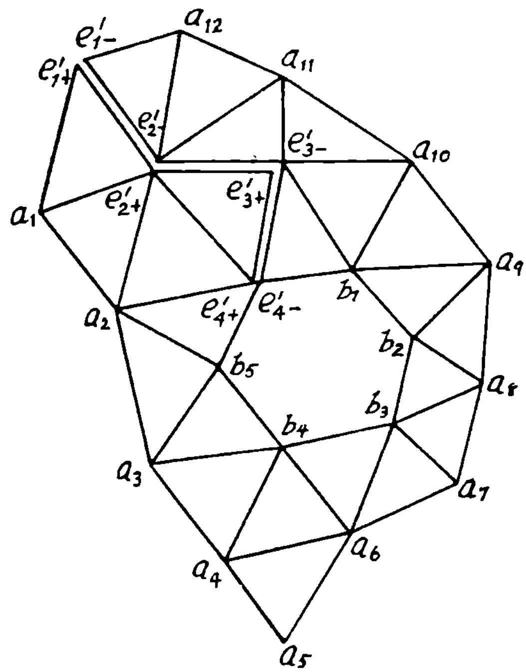


FIG. 40

But if one endpoint of the cross cut Λ lies on one contour and the other endpoint on another contour, K' has one less contour than K, so that the two contours

$$\langle e_1a_1 \cdots a_\mu e_1\rangle \quad \text{and} \quad \langle e_s b_1 \cdots b_\nu e_s\rangle$$

are replaced by the single contour (Fig. 40)

$$\langle e'_{1+}a_1 \cdots a_\mu e'_{1-} \cdots e'_{s-}b_1 \cdots b_\nu e'_{s+} \cdots e'_{1+}\rangle.$$

3.33. A closed interior two-sided cut replaces the closed broken line by two closed broken lines (Fig. 41)

$$\langle e'_{1+} \cdots e'_{s+}e'_{1+}\rangle \quad \text{and} \quad \langle e'_{1-} \cdots e'_{s-}e'_{1-}\rangle,$$

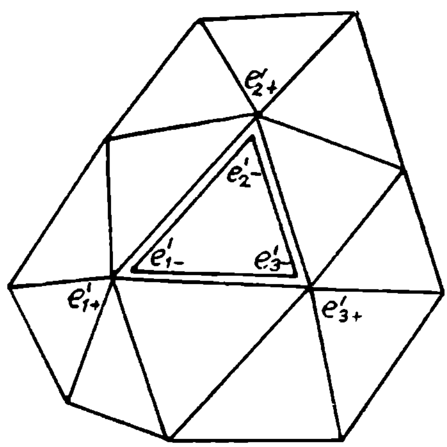


FIG. 41

which appear as two new contours; hence the number of contours increases by two. For a one-sided cut the closed broken line $\Lambda = \langle e_1 \cdots e_s e_1\rangle$ is replaced by the closed broken line

$$\langle e'_{1+} \cdots e'_{s+}e'_{1-} \cdots e'_{s-}e'_{1+}\rangle$$

which appears as a new contour of the surface K'; hence, K' has one more contour than K (Fig. 42).

In each of the complexes K and K' shown in Fig. 42 identically designated elements are to be identified. After the identification, the complex K is a triangulation of a Möbius band on which $\langle e_1e_2e_3e_1\rangle$ is a closed one-sided cut. This cut converts K into a complex K' isomorphic to a triangulation of a plane ring. The complex K has one contour $\langle ap_1p_2bq_1q_2a\rangle$; the complex K' has two contours:

$$\langle ap_1p_2bq_1q_2a\rangle \quad \text{and} \quad \langle e'_{1+}e'_{2+}e'_{3+}e'_{1-}e'_{2-}e'_{3-}e'_{1+}\rangle.$$

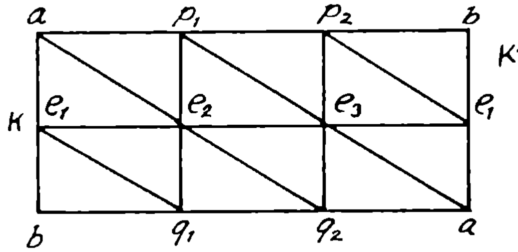


FIG. 42a

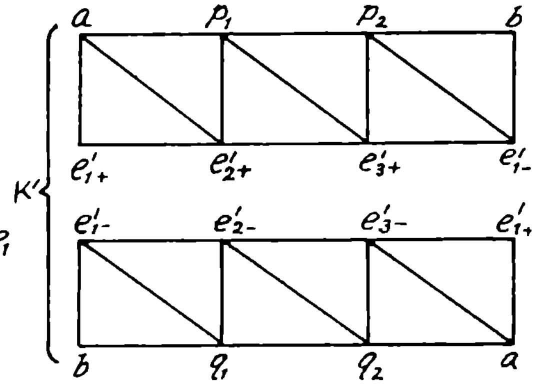


FIG. 42b

The reader is advised to construct a model of the complex K of the Möbius band out of paper and to cut it along the line $\langle e_1 e_2 e_3 e_1 \rangle$ with a pair of scissors.

REMARK. Both the open subcomplex $K \setminus \Lambda$ of K and the triangulation K', which results from cutting K, consist, as is easily seen, of the same number of mutually disjoint strongly connected complexes; if this number is equal to 1, we shall say that the cut does not separate the surface; if it is greater than 1, we shall say that the cut separates the surface and moreover, because of 3.22–3.25, into two components.

Hence

3.34. *Open cuts, cross cuts with endpoints on two different contours, and one-sided closed cuts do not separate a surface; closed two-sided cuts and cross cuts with endpoints in the same contour either do not separate a surface or separate it into two components.*

§3.4. Reduction of holes. Several different operations which decrease the number of contours of the surface by 1 or 2 are included under this heading. We shall discuss two types of reduction of a single hole and of a pair of holes.

1. REDUCTION OF THE FIRST KIND OF ONE HOLE. Consider a contour formed by a closed broken line with an even number of sides whose consecutive vertices are numbered as (Fig. 43)

$$a_1\,,\ c_1\,,\ c_2\,,\ \cdots\,,\ c_s\,,\ a'_1\,,\ c'_s\,,\ \cdots\,,\ c'_2\,,\ c'_1\,,\ a_1\,.$$

The vertices c_i and c'_i are identified for arbitrary i. As a result of this identification the closed broken line goes over into a simple broken line with endpoints a_1 and a'_1. An identification of this type represents an operation inverse to that of an open cut.

2. REDUCTION OF THE SECOND KIND OF ONE HOLE. Let

$$a_1 a_2 a_3 \cdots a_s a'_1 a'_2 a'_3 \cdots a'_s$$

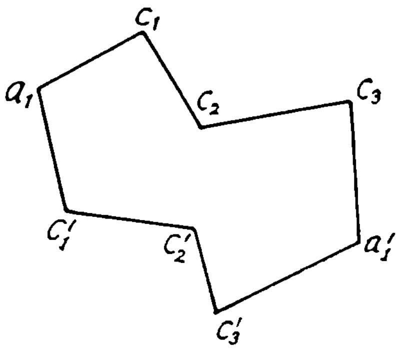


FIG. 43

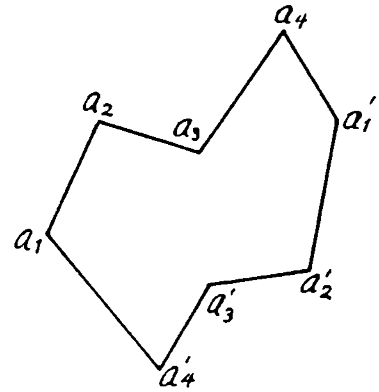


FIG. 44

be a contour (cf. Fig. 42b, where $a_1 = e'_{1+}$, $a_2 = e'_{2+}$, $a_3 = e'_{3+}$, $a'_1 = e'_{1-}$, $a'_2 = e'_{2-}$, $a'_3 = e'_{3-}$, or Fig. 44). The vertices a_i and a'_i are again identified. An identification of this type annihilates a one-sided cut on the surface.

Application of a reduction of the second kind to one of the two contours of a plane circular ring, or (what comes to the same) to one of the two bases of the lateral surface of a cylinder, yields a Möbius band (Fig. 45).

The proof is essentially the reverse of that given in connection with Fig. 42. For greater clarity let us carry out the above identification for another triangulation of a plane ring. (The identification itself is to be understood here and everywhere else in this chapter, as the combinatorial operation on a given, but arbitrarily chosen, triangulation defined in 3.1.)

The identification to be performed is indicated in Fig. 46a (elements de-

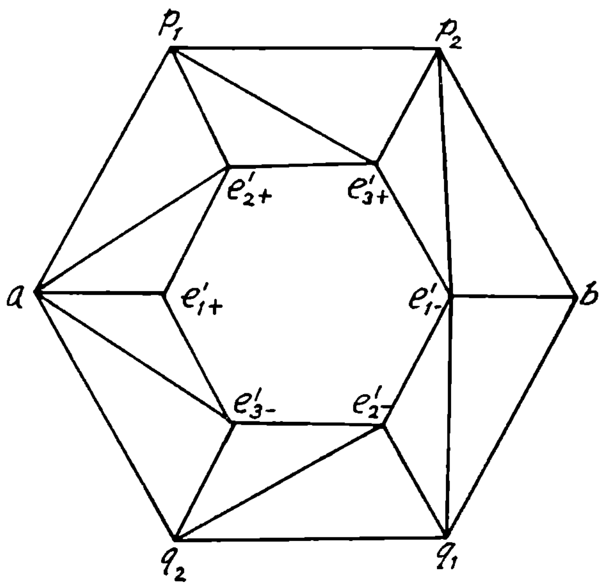
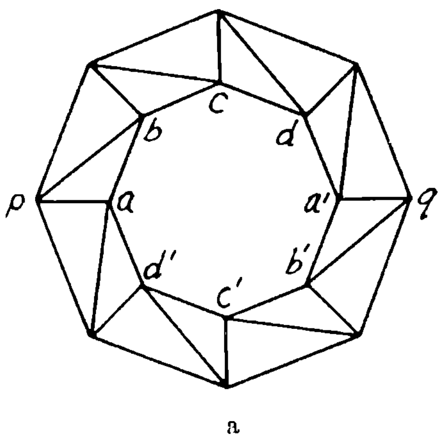


FIG. 45



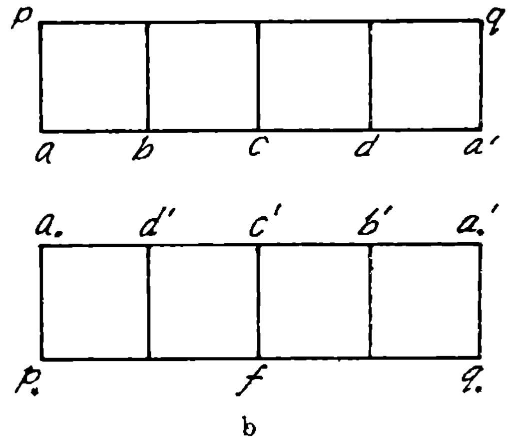
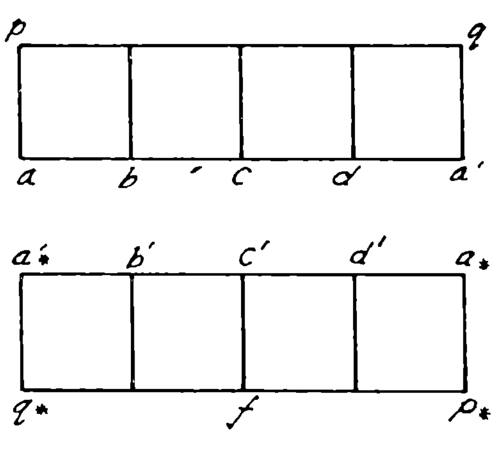


FIG. 46

noted by identical letters and having identical subscripts, even though provided with other marks which do not coincide, such as primes, asterisks, etc., are to be identified). Let us first cut the figure along the lines pa and $a'q$ (Fig. 46b), and then identify all elements denoted by the same letters and having the same subscripts. A preliminary rotation through 180° of the rectangle $a_*a'_*q_*p_*$ about the straight line $c'f$ has no effect on the result of this identification. [The rotation has to do only with the intuitive geometric interpretation of the given operation of identification and is completely unrelated to its combinatorial content, which is determined solely by the identification scheme. This remark also applies to analogous cases in the sequel (7.1, 7.2).]

The rotation transforms Fig. 46b into Fig. 47a, which in turn, after the indicated identifications, becomes Fig. 47b.

The one remaining identification of paq_* with p_*a_*q obviously converts the rectangle into a Möbius band. This proves the assertion.



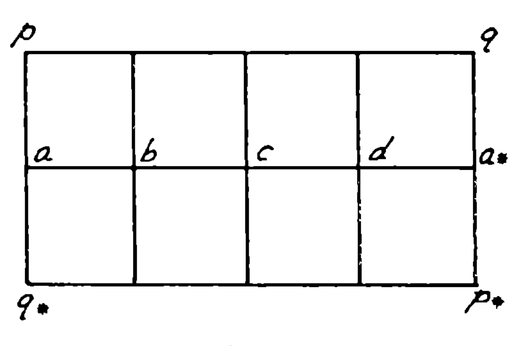


FIG. 47

This result clarifies the following proposition:

3.41. A reduction of the second kind of a hole with boundary Γ cut from a spherical surface (Fig. 48) gives the same result as closing up a somewhat larger hole with a Möbius band: the hatched ring is converted by an identification of the second kind of its inner boundary Γ into a Möbius band with boundary Γ'. (For convenience, a hemisphere with its equatorial plane, instead of a spherical surface, is shown in the figures. These are obviously equivalent topologically to a spherical surface; the letters in Fig. 48 refer to the innermost of the three circumferences.) We will, therefore, refer to a reduction of the second kind of one hole as a *closing up of a hole with a Möbius band.*

3. Reduction of Two Holes by Means of a Handle. This consists in pasting together the boundaries of the two holes. Let us consider this operation in detail in the case of two circular holes cut out of a sphere. Let the boundaries of these two holes be the closed broken lines

$$\langle a_1 \cdots a_s a_1 \rangle, \qquad \langle a'_1 \cdots a'_s a'_1 \rangle.$$

The identification may be effected in two ways.

The first case: handle of the first kind. Let a point a describe the contour Γ in any one direction, e.g., counterclockwise as seen from outside the sphere. Let the point a' of the contour Γ', which is to be identified with the point a, describe Γ' in the opposite sense (i.e., clockwise, as seen from the exterior of the sphere). In this case we say that the identification of Γ with Γ' is an *identification of the first kind* or that Γ and Γ' have been *fitted with a handle of the first kind* (Fig. 49). Now, as a describes Γ in a given sense (counterclockwise as seen from the outside of the sphere) let the point a' corresponding to a describe Γ' *in the same sense* (i.e., also counterclockwise). This is called an *identification of the second kind* or we say that Γ and Γ' have been *fitted with a handle of the second kind* (Fig. 50).

§4. Orientability of surfaces

§4.1. Definitions.

Definition 4.10. An oriented triangle is a triangle together with a prescribed sense of describing its boundary. Each of the two possible senses in

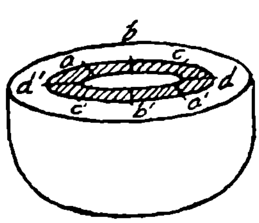


Fig. 48

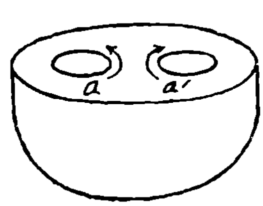


Fig. 49

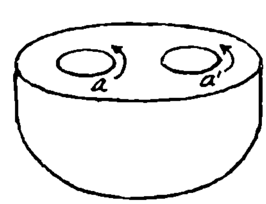


Fig. 50

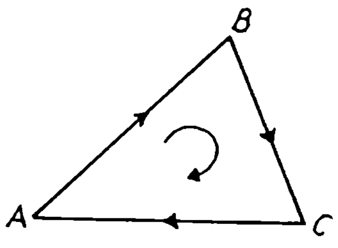


Fig. 51

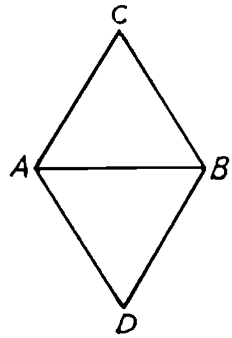


Fig. 52

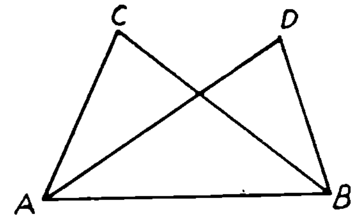


Fig. 53

which the boundary of the triangle may be described is called an *orientation* of the triangle. Hence a triangle ABC has two orientations, ABC and BAC. If one of the orientations of a triangle T^2 is denoted by t^2, the other will be denoted by $-t^2$. Each orientation of a triangle generates an orientation of each of its sides, the *induced orientation* of the sides: the orientation ABC induces on the sides AB, BC, AC the orientations AB, BC, CA (Fig. 51).

Consider two adjacent triangles (triangles with a common side) of a triangulation K of a surface and choose a definite orientation of these triangles. We shall say that the triangles are *coherently oriented* (*noncoherently oriented*) if they induce opposite (identical) orientations on their common side.

Thus, e.g., in Fig. 52 ABC and ABD are noncoherently oriented (since the orientations induced on AB by both triangles are the same, namely AB); but ABC and BAD are coherently oriented (since the orientations induced on AB are opposite).

Remark 1. In this definition it is essential to assume that the triangles are elements of the same triangulation: in the case shown in Fig. 53 (the figure is in the plane) the definition of coherent orientation of two triangles does not apply.

Definition 4.11. A triangulation K of a surface Φ is said to be *orientable* if it is possible to orient all the triangles of K in such a manner that every two adjacent triangles are coherently oriented. In the contrary case the triangulation K is said to be *nonorientable*.

Definition 4.111. Let K be an orientable triangulation. Then it is possible to choose orientations $t^2_1, \cdots, t^2_p$ for all the triangles $T^2_1, \cdots T^2_p$ of K in such a way that every pair of adjacent triangles is coherently oriented. A set of orientations $t^2_1, \cdots, t^2_p$ of all the triangles $T^2_i \in K$ satisfying this condition is called an *orientation of the triangulation K*.

If an orientation t^2_i of a triangle T^2_i of K is given, the requirement that every pair of adjacent triangles be coherently oriented defines uniquely

first the orientations of all the triangles adjacent to T^2_i and then step by step, by virtue of the strong connectedness of the triangulation K, the orientations of all the remaining triangles of K. Hence, for any triangle T^2_i of an orientable triangulation K and for any orientation t^2_i of T^2_i, there is a unique orientation of all the triangles of K which contains the given orientation t^2_i of T^2_i.

Therefore,

4.112. *Every orientable triangulation has precisely two orientations.*

Remark 2. Defs. 4.11 and 4.111, as well as Theorem 4.112, may be applied not only to triangulations of surfaces but also to all open strongly connected subcomplexes of these triangulations, e.g., to cyclic and semi-cyclic stars.

Let Q be a cyclic or semi-cyclic star. The complex Q is obviously isomorphic to a plane complex Q'. An orientation of every triangle of Q', e.g., counterclockwise, yields an orientation of Q', and hence an orientation of Q.

Hence

4.12. *Cyclic and semi-cyclic stars are orientable.*

In this connection we have the following proposition (Fig. 54):

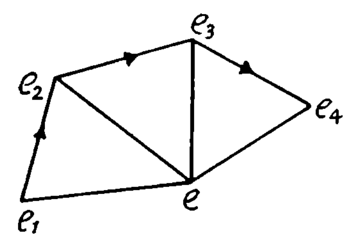


Fig. 54

4.121. Let Q be the semi-cyclic star of a vertex e; let (e_1e) and (ee_s) be bounding segments of this star [i.e., segments each of which borders on a single triangle (e_1ee_2) and $(e_{s-1}ee_s)$, respectively, of the star Q]. Then each orientation of Q induces orientations on the segments (e_1e) and (ee_s) which are extensions of each other [i.e., the orientations (e_1e) and (ee_s) or the orientations (ee_1) and (e_se)].

It is not difficult to prove this proposition by, e.g., complete induction on the number of triangles in Q.

In this chapter we shall use without proof the following proposition which will be proved in Chapter X (naturally, independently of the results of the present chapter).

4.13. *Let K and K' be arbitrary triangulations of the same surface or of two homeomorphic surfaces; if one of these triangulations is orientable, the other is also orientable.*

This theorem is called the *theorem on the invariance of orientability*; in the light of 4.13, it is natural to make the following definition:

4.14. A surface Φ is said to be *orientable* if any (arbitrary) triangulation of Φ is orientable. In the contrary case, the surface is said to be *nonorientable.*

4.13 implies

4.15. *If a surface is orientable (nonorientable), then every surface homeomorphic to the given surface is orientable (nonorientable).*

DEFINITION 4.16. Let K be a triangulation of a surface Φ. A sequence of oriented triangles T^2_1, T^2_2, $\cdots$, T^2_s of this triangulation with orientations t^2_1, $\cdots$, t^2_s is called a *disorienting sequence* if the following conditions are satisfied:

For $i = 1, \cdots, s - 1$, the triangles T^2_i and T^2_{i+1} adjoin each other and are coherently oriented; the triangles T^2_s and T^2_1 also adjoin each other but are noncoherently oriented.

If K is an orientable triangulation, it is impossible to construct a disorienting sequence composed of oriented triangles of K. Indeed, if z is that orientation of K which contains t^2_1, it is easy to see that the orientations t^2_2, $\cdots$, t^2_s are also contained in z; but then the orientation z would contain two noncoherently oriented adjoining triangles t^2_1 and t^2_s. Conversely, *if a triangulation K is nonorientable, it contains a disorienting sequence.* In fact, choose any triangle T^2_1 of K and give it any orientation t^2_1. Orient every triangle adjoining T^2_1 coherently with respect to t^2_1. In general, orient all the remaining triangles one by one in such a way that a triangle adjoining an already oriented triangle receives an orientation coherent

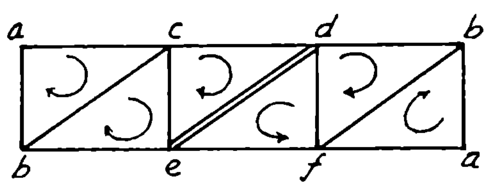


FIG. 55

with that of its already oriented neighbor. Since K is, by assumption, nonorientable, a continuation of this process will eventually yield a triangle T^2_s which receives two opposite orientations depending on which of the already oriented neighbors of the triangle T^2_s is used for the definition of the orientation of T^2_s itself. In other words, it is possible to go from T^2_1 to T^2_s, passing through consecutive triangles, by two paths, which lead to opposite orientations of T^2_s.

Going from T^2_1 to T^2_s along the first path and then from T^2_s to T^2_1 by the second, we obtain a disorienting sequence of triangles.

Hence, we have proved

4.17. *In order that a triangulation be nonorientable, it is necessary and sufficient that it contain at least one disorienting sequence of triangles.*

EXAMPLE. Let us consider the triangulation of the Möbius band shown in Fig. 55. Giving the triangles of this triangulation the orientations designated by arrows in Fig. 55, we obtain a disorienting sequence. Hence, the Möbius band is a nonorientable surface.

Theorem 4.17 implies that if a subcomplex K_0 of a triangulation K is

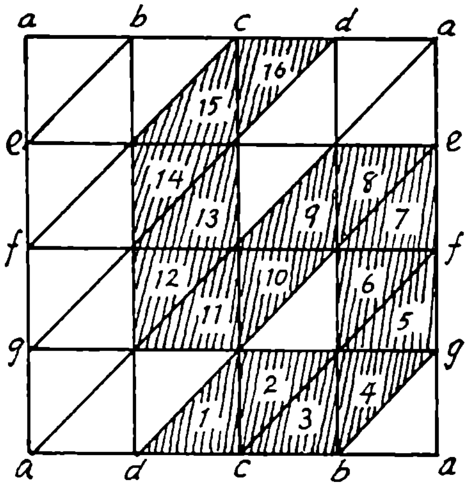


FIG. 56

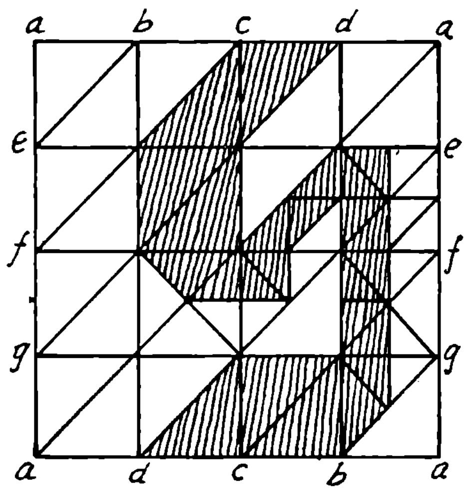


FIG. 57

nonorientable, then K is nonorientable. Hence, it follows in turn that if a triangulation K contains as a subcomplex any triangulation of the Möbius band, then K is a nonorientable triangulation. It is easy to see that the triangulations of the Klein bottle and the projective plane given above contain triangulations of Möbius bands. Consequently, both the Klein bottle and the projective plane are nonorientable surfaces. It is easy to convince oneself of this immediately.

Let K be a triangulation of a nonorientable surface.

That part of $\| K \|$ which is covered by the triangles of a disorienting sequence always contains a Möbius band subdivided into triangles of some regular subdivision (it suffices to take a second order barycentric subdivision) of K. A disorienting sequence is shown in Fig. 56; Fig. 57 shows a regular subdivision of the triangles of this sequence; the hatched triangles form a Möbius band.

Hence:

4.18. *In order that a surface Φ be nonorientable, it is necessary and sufficient that some triangulation K_1 of the surface Φ contain a subcomplex which is a triangulation of a Möbius band.* A regular subdivision of an arbitrary triangulation K of the surface Φ can be taken as the triangulation K_1.

REMARK. It would be possible to prove that every surface containing a subset homeomorphic to a Möbius band is nonorientable. The converse of this assertion is contained in what has just been proved.

We shall prove the following fundamental proposition:

4.19. *In order that a surface Φ be nonorientable, it is necessary and sufficient that some triangulation K of Φ contain a one-sided cut line.*

Proof. Suppose that K contains a one-sided cut line

$$\Lambda = \langle e_1 e_2 \cdots e_s e_1 \rangle.$$

Let us consider the closed chain of semi-stars

$$Oe^+_1, \cdots, Oe^+_s, Oe^-_1, \cdots, Oe^-_s, Oe^+_1$$

[see (3.242)]. Let us form the closed chain (Fig. 58)

$$T_1 = (e_se_1a), \cdots, T_k = (ce_{s-1}e_s), \cdots, T_r = (de_se_1)$$

consisting of the triangles of the semi-stars $Oe^+_1, \cdots, Oe^+_s, Oe^-_1$. Let us orient the semi-star Oe^+_1 in any way and let the induced orientation on (e_1e_2) be, e.g., $(e_1e_2)^{\rightarrow}$. The orientation of the semi-star Oe^+_1 defines, in particular, the orientation of the triangle contained in both semi-stars Oe^+_1 and Oe^+_2, and consequently of the whole semi-star Oe^+_2. Hence the orientations of all the semi-stars $Oe^+_1, \cdots, Oe^+_s, Oe^-_1$ are defined step by step.

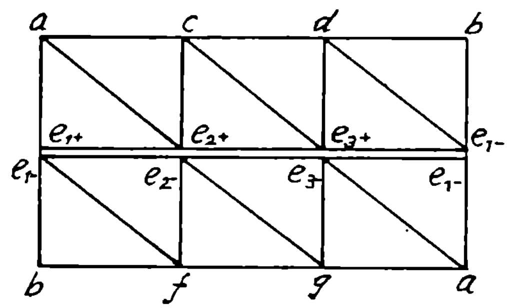


FIG. 58

Moreover, all these orientations induce on the broken line Λ the same direction which, for the chosen orientation of Oe^+_1, is the sense

$$\langle e_1e_2 \cdots e_se_1\rangle^{\rightarrow}.$$

Every pair of triangles T_i and T_{i+1}, $i = 1, 2, \cdots, s - 1$, have been oriented coherently. However, since $T_1 = (e_se_1a)$ and $T_r = (de_se_1)$ have been oriented so that the orientations induced on their common side by both triangles are the same, i.e., $(e_se_1)^{\rightarrow}$, the triangles T_1 and T_r are noncoherently oriented. The chain of triangles

$$T_1, \cdots, T_r$$

with the above orientations forms a disorienting sequence, so that K is a nonorientable triangulation.

Conversely, if K is a nonorientable triangulation, some regular subdivision of K contains a triangulation of the Möbius band which in turn contains a one-sided cut. This proves 4.19.

§5. The connectivity of a surface. Euler's theorem

§5.1. Throughout this section K will denote a triangulation of a surface Φ.

5.11. Let us denote by ρ_0, ρ_1, ρ_2 the number of vertices, edges, and triangles, respectively, of a triangulation K of a surface Φ; the number

$$\chi(K) = \rho_0 - \rho_1 + \rho_2$$

is called the *Euler characteristic of the triangulation.*

In this chapter we shall accept without proof the following theorem (proved in Chapter X).

THEOREM 5.12. (INVARIANCE OF THE EULER CHARACTERISTIC.) *If K and K' are triangulations of the same surface or of two homeomorphic surfaces, the Euler characteristics of K and K' are equal.*

In virtue of Theorem 5.12 it is natural to call the Euler characteristic of any triangulation of a surface the Euler characteristic of the surface.

EXERCISE. Show (by considering any triangulation of the corresponding surface) that the Euler characteristic of the sphere and the projective plane is equal to 2 and 1, respectively, and that the Euler characteristic of a plane ring (cylinder), Möbius band, torus, and Klein bottle is equal to zero.

REMARK 1. In this chapter we shall say that a one-dimensional closed subcomplex L of a triangulation K does not separate K, if the open subcomplex $K \setminus L \subset K$ is a strongly connected complex.

DEFINITION 5.13. *The connectivity of a triangulation* K is the maximum integer k for which there exists a closed one-dimensional subcomplex of connectivity k which does not separate K. The connectivity of a triangulation K will be denoted by $q(K)$.

REMARK 2. It follows easily from the Jordan theorem that every elementary curve of positive connectivity separates the sphere.

This implies, without the use of any invariance theorem, that the connectivity of an arbitrary curved triangulation of a sphere, and consequently of every surface homeomorphic to the sphere, is equal to zero. The reader is advised to prove this assertion.

EULER'S THEOREM 5.14.

$$\chi(K) = 2 - q(K). \tag{5.14}$$

Proof. Let L be a one-dimensional nonseparating closed subcomplex of K of connectivity $q = q(K)$. Let us enumerate the triangles of K in an order $T^2_1, \cdots, T^2_r$, where $r = \rho_2$, such that the triangle T^2_{i+1}, $i = 1, 2, \cdots, \rho_2 - 1$, will have a side T^1_i not contained in L in common with one of the triangles $T^2_1, \cdots, T^2_i$ (this can be done, since L, by hypothesis, does not separate K).

Let us delete the triangle T^2_1 from $K = K_0$ and denote the remaining subcomplex of the triangulation K by K_1; next delete both the triangle T^2_2 and the segment T^1_1 and denote the remaining complex by K_2, etc., until

all ρ_2 triangles are exhausted. We obtain complexes $K_1, K_2, \cdots, K_{r-1}$, where T^2_r is the only 2-element of the subcomplex K_{r-1}; finally, $K_r = K_{r-1} \setminus T^2_r$ is a 1-complex.

We shall prove that all the complexes K_i are connected. For $K_0 = K$ this is true by hypothesis. Assuming that K_i is connected, we shall show that the complex $K_{i+1} = K_i \setminus T^2_{i+1} \setminus T^1_i$ is also connected. It suffices to prove that every pair of vertices a, b of K_{i+1} can be connected by a broken line in K_{i+1}. Let $[ab]$ be a broken line connecting the vertices a and b in K_i. If this broken line is not contained in K_{i+1}, one of its links is the side T^1_i of the triangle T^2_{i+1}. If we prove that the other two sides T^1_1 and T^1_2 of of the triangle T^2_{i+1} are contained in K_{i+1}, then the link T^1_i of $[ab]$ can be replaced by a broken line of two links consisting of T^1_1 and T^1_2 and the assertion will be proved. But the side T^1_1 (or T^1_2) does not appear in K_{i+1} only if it is a side T^1_{h-1} of some triangle T^2_h, $h < i + 1$. This, however, cannot be, since, if the segment T^1_1, which is a side of T^2_{i+1}, is also a side T^1_{h-1}, $h < i + 1$, of the triangle T^2_h, then it would have to be a side of yet a third triangle T^2_k, $k < h$, which is impossible. This proves that K_{i+1} is connected. The complex K_r does not separate K since all the triangles of K were enumerated in the form of a chain not intersecting K_r; since $K_r \supseteq L$, Theorem 1.23 implies that $\pi^1(K_r) \geq \pi^1(L) = q$, and consequently, by the definition of L and q, $\pi^1(K_r) = \pi^1(L) = q$.

The 1-complex K_r has ρ_0 vertices and $\rho_1 - (\rho_2 - 1)$ edges (since $\rho_2 - 1$ edges have been deleted from K). Hence, by (1.24),

$$\rho_0 - [\rho_1 - (\rho_2 - 1)] = \pi^0(K_r) - \pi^1(K_r) = 1 - q,$$

or $\rho_0 - \rho_1 + \rho_2 = 2 - q$, q.e.d.

5.12 and 5.14 imply

5.15. THEOREM ON THE INVARIANCE OF CONNECTIVITY. *Two arbitrary triangulations of two homeomorphic surfaces have the same connectivity.*

It is therefore natural to call the connectivity of any triangulation of a given surface the *connectivity of the surface.*

§6. Simple surfaces

§6.1. Closed cuts. In this and in the following sections a triangulation will mean a triangulation of some surface, and a *closed cut line* of a triangulation K will mean any simple closed broken line Λ consisting of elements of the triangulation K and such that at least one of its segments is an interior element of the triangulation K (i.e., one which is not situated on its boundary).

If the closed cut line Λ contains more than one vertex belonging to the boundary of K, then Λ, as is easily seen, consists of one or more cross cuts

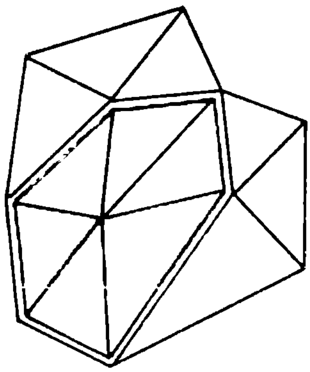


FIG. 59

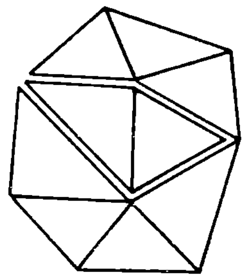


FIG. 60

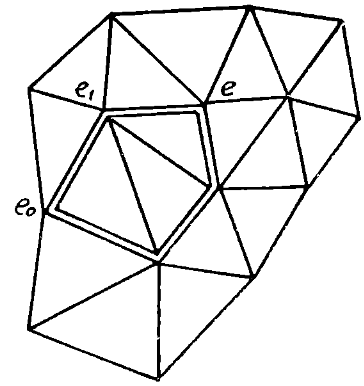


FIG. 61

and of vertices, and perhaps arcs, which lie on the boundary of K (Λ may or may not contain such arcs; see Figs. 59–60).

In this case the cut operation itself is realized by the cross cuts contained in Λ.

But if Λ has a single element, a vertex e_0 (Fig. 61), on the boundary, then the cut operation along the line Λ is defined as follows: take any second vertex e of Λ and perform an open cut along one of the two arcs $[e_0e] \subset \Lambda$, say along the arc $[e_0e_1e]$, and then a cross cut along the second arc $[e_0e]$ contained in Λ.

REMARK. In cases where there can be no misunderstanding, instead of "cut line" we shall simply say "cut".

The following proposition will be required in the sequel:

6.11. If a triangulation K contains a nonseparating closed cut Λ some of whose elements lie on the boundary, there is a regular subdivision K' of K which contains a closed cut Λ' which does not separate K', and such that *all* its elements are interior elements.

Passing to the proof of 6.11, we note first that:

In order that a closed cut Λ not separate the triangulation K, it is sufficient that two triangles T_1 and T_2 having a common side contained in Λ can be joined by a chain of triangles in which every two consecutive triangles have a common side not contained in Λ. If this is the case, we shall say that T_1 and T_2 are joined by a chain of triangles which avoids the cut Λ.

We shall show that, passing, if necessary, to a barycentric subdivision, it can always be assumed that the "stars" of the vertices e_i and e_j, belonging to a contour Γ, can intersect only if the edge (e_ie_j) is in Γ. Indeed, let K be a triangulation and K' its barycentric subdivision. It is easily seen that there are in K' no interior edges both of whose endpoints lie on the boundary of the surface. It follows that the triangulation K' has the following property: the "stars" lying on the boundary of the surface with

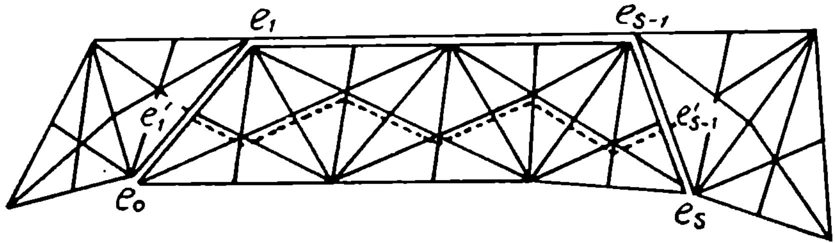


FIG. 62

vertices e' and e'' intersect only if both vertices e' and e'' belong to a single contour Γ and are neighboring vertices on this contour. We shall assume that the triangulation K has this property to begin with.

To obtain the line Λ' it is necessary to remove from the boundary every piece Λ_0 of the cut Λ which lies on the boundary of the surface (Fig. 62). This is achieved as follows: take a barycentric subdivision K' of the triangulation K; next, consider the part $[e'_1 e_1 \cdots e_{s-1} e'_{s-1}]$ of the cut, consisting of the piece $\Lambda_0 = [e_1 \cdots e_{s-1}]$ lying on the boundary of the surface, supplemented by the two interior edges $(e'_1 e_1)$ and $(e_{s-1} e'_{s-1})$; let us replace this part of the cut by the broken line $\Lambda_1 = [e'_1 \cdots e'_{s-1}]$ whose vertices are interior vertices of the triangulation K', lying on the boundary of the "stars" of the vertices of the piece Λ_0 (in the triangulation K').

Let us prove that this replacement yields a simple closed broken line Λ' which does not separate K'. Indeed, the broken line Λ_1, as is easily seen, is a simple nonclosed broken line lying in the interior of the triangulation K'; in addition, all the vertices lying on Λ_1, with the exception of e'_1 and e'_{s-1}, do not lie on Λ. Therefore, the broken line Λ' is the union of two simple nonclosed broken lines having only their endpoints in common. Hence Λ' is a simple closed broken line.

It remains to be shown that the broken line Λ' does not separate the triangulation K'. This assertion in turn is an easy consequence of the following. There are two triangles in K' with a common side in Λ_1, which can be joined by a chain of triangles avoiding Λ'.

To prove this last assertion, let us consider a chain of triangles $T'_1, \cdots, T'_s$ of K' connecting the triangles T'_1 and T'_s with common side $(e'_1 e_1)$ and avoiding Λ. The chain will contain at least one triangle with a side in Λ_1, and vertices on Λ_0; indeed, one (and only one) of the triangles T'_1, T'_s, say T'_1, is a triangle of this sort. Let T'_k be the last triangle of the chain with the indicated property (obviously $k \neq s$). It is then easy to see that T'_k and T'_{k+1} have a common edge contained in Λ_1 and can be connected by a chain of triangles avoiding Λ'.

§6.2. Definition of simple triangulations. Invariance under regular subdivisions.

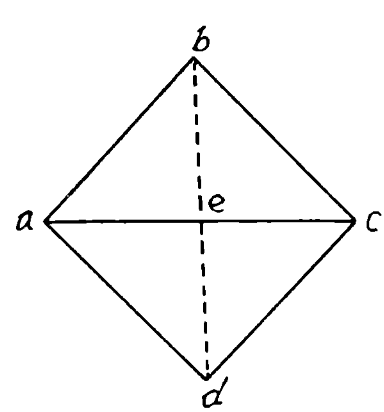
DEFINITION 6.21. A triangulation of a surface Φ is said to be *simple* if it

is separated by an arbitrary closed cut. An example of a nonsimple triangulation is a triangulation of a Möbius band or of a torus. It can be shown that: *If a triangulation of a surface Φ is simple, then every triangulation of every surface homeomorphic to Φ is also simple.*

However, we shall need merely:

6.22. *Any regular subdivision of a simple triangulation is simple.*

Proof. It suffices to deduce from the existence of a nonseparating cut of a subdivision K' of the triangulation K the existence of a nonseparating cut in K. Let K' be an elementary subdivision with respect to an interior side (in the case of a side lying on the boundary, the reasoning which follows below is even simpler) (ac) which borders on the triangles (abc) and (adc) and denote the new vertex on (ac) by e. If Λ is a nonseparating cut in K', which does not appear as such in K, Λ contains at least one of the segments (be), (ed). If Λ contains only one of these segments and goes through, e.g., $[bec]$, then replacing the broken line $[bec]$ in Λ by the segment (bc) yields a nonseparating cut in K.



Let Λ contain both (be) and (ed). We shall consider two cases:

1°. Let one of the points a, c, say c, be contained in either Λ or a contour Γ of the surface containing at least one vertex of the cut Λ. The triangle (ade) can be connected, avoiding the cut Λ, with one of the triangles (bec), (dec) by a chain which does not contain the other triangle. If the triangles joined are (ade) and (bec), the segment (bc), the part of Γ up till the first vertex of Λ, and the part of Λ not containing $[deb]$, form a nonseparating cut [our assumptions imply that the side (bc) is necessarily an interior side; otherwise $[bd]$ would separate (aed) from (bec)] of the original triangulation.

2°. Suppose that neither a nor c is contained in Λ. Either, one of the points a, c, say c, is an interior point, or the contour Γ which contains this point has no point of Λ. Then, replacing the segment $[bed]$ by the broken line $[bcd]$ in Λ, again yields a nonseparating cut in K.

6.23. *Every triangulation K_0 which is a subcomplex of a simple triangulation is simple.*

Indeed, every closed cut Λ in K_0 is a closed cut in K. Let T^2_1 and T^2_2 be two triangles of K_0 with a common side contained in Λ. Since it is impossible to join them in K by a chain of triangles avoiding Λ, it is all the more impossible to join them by such a chain in K_0, which was to be proved.

§6.3. Elementary lemmas. We shall require the following propositions in the sequel. Their proof is left to the reader.

LEMMA 6.31. A topological mapping of the circumference S of a circle Q onto the circumference S' of a circle Q' can be extended to the whole circle (i.e., there exists a topological mapping of the circle Q onto the circle Q' which maps corresponding points of the circumferences into each other).

Hint. Use polar coordinates.

LEMMA 6.32. A topological mapping of an arc $[ab]$ of the circumference S of a circle Q onto the diameter $[cd]$ of a semi-circle Q_1 can be extended to a topological mapping of the whole circle Q onto all of the semi-circle Q_1 in such a way that the circumference of the circle is mapped onto the boundary of the semi-circle.

REMARK. Let us call every compactum Q which is the topological image of an ordinary circle Q_0 a *topological circle*. It will be shown in Chapter V that *every* topological mapping of Q_0 onto Q maps the circumference of Q_0 onto the same simple closed curve $S \subset Q$, called the *topological circumference of* Q. If this assertion is accepted, Lemmas 6.31 and 6.32 obviously remain true if Q and S are taken to mean a topological circle and its topological circumference.

§6.4. Classification of simple surfaces. *A surface is said to be simple* if it has at least one simple triangulation.

THEOREM 6.41. *Every simple surface whose boundary consists of a single contour is homeomorphic to a circle.*

Proof. As usual, let us denote by ρ_2 the number of triangles of a (simple) triangulation K of the surface Φ. The theorem is obviously true for $\rho_2 = 1$. Assuming the theorem true for $\rho_2 \leq n$, we shall prove it for $\rho_2 = n + 1$. The triangulation K contains a triangle which adjoins the boundary of the surface Φ. Let $T^2 = (e_0e_1e_2)$ be such a triangle.

Three cases may arise:

a) All three vertices, but only two of the sides, (e_0e_2) and (e_2e_1), of the triangle T^2 are situated on the boundary of the surface Φ.

b) All three vertices, but only one side, (e_0e_1), of the triangle T^2 are on the boundary of the surface Φ.

c) Two vertices, e_0 and e_1, and one side (e_0e_1) of the triangle T^2 are on the boundary of the surface Φ.

In cases a) and c) the closed cut $\langle e_0e_1e_2\rangle$, i.e., in case a) the cross cut $\Lambda = [e_0e_1]$, and in case c) the cross cut $\Lambda = [e_0e_2e_1]$ (Figs. 63–65) separates K into two disjoint triangulations. One of them is the triangle T^2 with its sides and vertices and the other is some triangulation K_1. The boundary of K_1 is a simple closed broken line consisting of Λ and the piece $[e_0ce_1]$ of

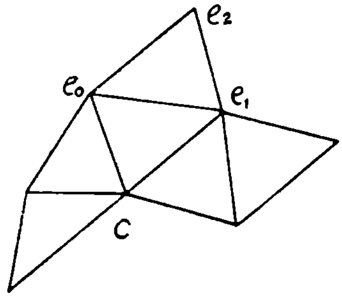


FIG. 63 (Case a)

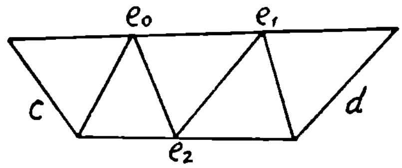


FIG. 64 (Case b)

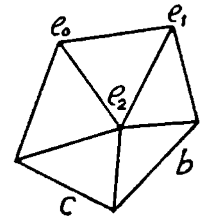


FIG. 65 (Case c)

the boundary of K. The triangulation K_1 is simple by virtue of Theorem 6.23 and contains n triangles. Consequently, the surface $\| K_1 \|$ can be mapped topologically onto a semi-circle Q_1 of a circle Q in such a way that Λ maps onto the diameter of the semi-circle. Mapping T^2 onto the other half circle Q_2 of the circle Q in such a way that this mapping coincides with the preceding one on Λ yields a mapping of all of the surface Φ onto the circle Q.

In case b) the points e_0, e_1, and e_2 separate the boundary of the surface Φ into three pieces $\Lambda_1 = [e_0ce_2]$, $\Lambda_2 = [e_2de_1]$, and $\Lambda_3 = [e_1e_0]$. The closed cut along the curve consisting of Λ_1 and the side (e_2e_0) of the triangle T^2, i.e., the cross cut along (e_0e_2), separates K into two disjoint simple triangulations K_1 and K_2, each of which contains no more than n triangles. Therefore, the surfaces $\| K_1 \|$ and $\| K_2 \|$ are homeomorphic to a circle. Mapping each of them onto the two half circles of the same circle in such a way that both mappings coincide on (e_0e_2), yields a mapping of the surface $\| K \|$ onto a circle, which was to be proved.

THEOREM 6.42. *A simple closed surface is homeomorphic to the sphere.*

Proof. Deleting one triangle T^2 from a simple triangulation K of the closed surface Φ yields a simple triangulation K_1 with one contour. Mapping $\| K_1 \|$ and $\| T^2 \|$ topologically onto two hemispheres in such a way that both mappings coincide on the boundary of the triangle T^2 and take this boundary onto the equator of the sphere, we obtain the required mapping of the whole surface $\| K \|$ onto the sphere.

EXERCISE. Using reasoning analogous to that in the proof of Theorem 6.41, prove without the invariance theorem, that the Euler characteristic of an arbitrary triangulation of a surface homeomorphic to the circle is equal to 1. From this derive (also without using the invariance theorem) the classical theorem of Euler:

The Euler characteristic of an arbitrary triangulation of a surface homeomorphic to the sphere is equal to 2.

REMARK 1. The Jordan theorem implies that every triangulation of the sphere is simple. Hence:

If one triangulation of a closed surface Φ is simple, Φ is homeomorphic to the sphere, and then every triangulation of Φ is simple.

DEFINITION 6.43. A sphere with r circular holes whose boundaries are disjoint by pairs is called a *normal simple surface* Q_r with r contours. The boundaries of the holes are part of the surface.

REMARK 2. Since a topological mapping of one surface Φ onto another Φ' maps the boundaries of the surfaces Φ and Φ' onto each other (this assertion will be proved, independently of the results of this chapter, in Chapter V), and homeomorphic sets (in particular, elementary curves) have the same number of components, homeomorphic surfaces have the same number of contours.

REMARK 3. To cut a circular hole out of the sphere is equivalent to cutting a spherical sector out of it.

REMARK 4. The surfaces Q_r defined in 6.43 are obviously homeomorphic to the surfaces obtained by removing from a circle Q the interior of $r - 1$ mutually disjoint circles contained in the interior of Q. The surfaces Q_r will be shown in just this way in the figures.

THEOREM 6.44. *Every simple surface Φ with r contours is homeomorphic to a normal simple surface Q_r.*

This theorem is obviously included in

6.45. *Let Φ be a surface with r contours which has a simple triangulation K. Let C be a topological mapping of one of the contours of the surface Φ onto one of the contours of a normal surface Q_r. The mapping C can be extended to a topological mapping of the whole surface Φ onto the surface Q_r* (taking, in virtue of the invariance of the boundary, the boundary of Φ into the boundary of Q_r).

Proof of Theorem 6.45. Theorem 6.41 and Lemma 6.31 imply that Theorem 6.45 is true for $r = 1$. We shall assume Theorem 6.45 true for $r \leq n$ and prove it for $r = n + 1$.

Thus, let K be a simple triangulation of a surface Φ with $n + 1$ contours, among which we mark any two contours, Γ_1 and Γ_2. Let the surface Q_{n+1} be represented by a definite triangulation (in order to make it possible to speak of cuts on the surface). Let us single out two arbitrary contours Γ'_1, Γ'_2 from among the contours of Q_{n+1}. Let Λ be a simple broken line in K joining Γ_1 and Γ_2 and let Λ' be a broken line on Q_{n+1} joining Γ'_1 and Γ'_2. Cuts (Fig. 66) along the broken lines Λ and Λ' transform the surfaces Φ and Q_{n+1} into $\Phi_1 = \| K_1 \|$ and $Q_{n+1,1}$, respectively, where each pair of contours: Γ_1, Γ_2 and Γ'_1, Γ'_2, is replaced by the single contour

$$\Gamma = L_1 \cup \Lambda^+ \cup L_2 \cup \Lambda^-,$$

and

$$\Gamma' = L'_1 \cup \Lambda'^+ \cup L'_2 \cup \Lambda'^-,$$

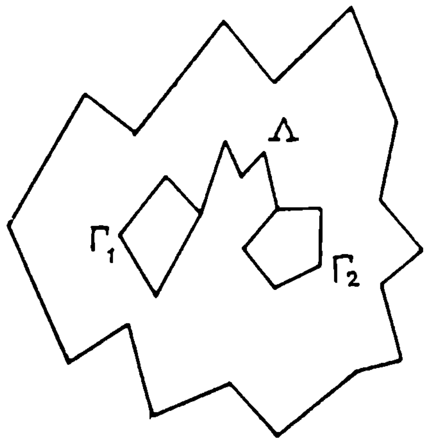


FIG. 66

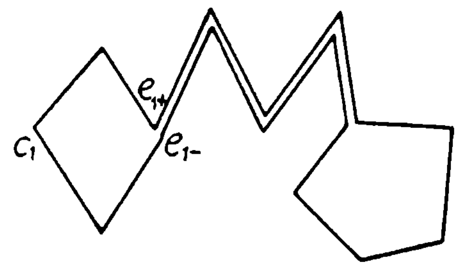


FIG. 67

respectively. Here, L_1, L_2, etc., are simple arcs arising from Γ_1 and Γ_2, etc., after the cut (for example, in Fig. 67 L_1 is the simple arc $[e_{1-}c_1e_{1+}]$). The surfaces Φ_1 and $Q_{n+1,1}$ have only n contours. The surface $Q_{n+1,1}$ can obviously be identified with the surface Q_n.

We shall show that Φ_1 is a simple surface. Indeed, if the triangulation K_1, into which the triangulation K is transformed after the cut, were to contain a nonseparating closed cut, some regular subdivision K'_1 of K_1 would contain a nonseparating closed cut D'_1 without boundary elements. Under the identification which returns K_1 into K, the subdivision K'_1 of K_1 goes over into some regular subdivision K' of K and the nonseparating cut D'_1 of K'_1 into a nonseparating cut of K'. But this is impossible, since K', being a regular subdivision of the simple triangulation K, is simple, by Theorem 6.22.

Now let $C(\Gamma_1, \Gamma'_1)$ be a topological mapping of the curve Γ_1 onto Γ'_1. It defines a topological mapping $C(L_1, L'_1)$ of the simple arc L_1 onto the simple arc L'_1, which maps both endpoints e_{1+}, e_{1-} of the arc L_1 onto the endpoints of the simple arc L'_1. The mapping $C(L_1, L'_1)$ can be extended to a mapping $C(\Gamma, \Gamma')$ of the closed curve Γ onto Γ', which maps Λ^+, L_1, Λ^-, L_2 onto Λ'^+, L'_1, Λ'^-, L'_2, respectively, and which maps two points of the arcs Λ^+ and Λ^- lying opposite each other into two points of Λ'^+ and Λ'^- lying opposite each other.

By the inductive hypothesis, the topological mapping $C(\Gamma, \Gamma')$ can be extended to a topological mapping $C(\Phi_1, Q_n)$ of the surface Φ_1 onto $Q_{n+1,1}$, i.e., onto Q_n. If now an identification which annuls the cuts just made is performed, the homeomorphism $C(\Phi_1, Q_n)$ between Φ_1 and $Q_{n+1,1}$ goes over into a homeomorphism $C(\Phi, Q_{n+1})$ between Φ and Q_{n+1} which coincides on Γ_1 with the originally given homeomorphism $C(\Gamma_1, \Gamma'_1)$. This proves Theorem 6.45 and hence 6.44.

6.44 and Remark 2 imply:

6.440. *Two simple surfaces are homeomorphic if, and only if, they have the same number of contours.*

Let us now prove that *the Euler characteristic of a simple surface with r contours is* $2 - r$. Because of the invariance of the Euler characteristic, it suffices to prove that

$$\chi(Q_r) = 2 - r. \tag{6.46}$$

Since the Euler characteristic of the sphere is 2 and because of the invariance of the Euler characteristic, the equality $\chi = \chi(Q_r) = 2 - r$ follows from the following obvious proposition:

Deleting from a triangulation K an arbitrary triangle and retaining its vertices and sides decreases the Euler characteristic of the triangulation by one.

EXERCISE. Prove by means of the Jordan theorem that *every* triangulation of a simple surface is simple. For the proof it suffices to consider arbitrary curvilinear triangulations of simple normal surfaces.

§7. Classification of closed surfaces

§7.1. Genus of a surface. Normal surfaces of a given genus.

DEFINITION 7.11. Let Φ be a closed surface. *The genus of the surface Φ* is, by definition, one half its connectivity if Φ is orientable, and its connectivity decreased by 1 if Φ is nonorientable. The genus of a closed surface Φ is denoted by $p(\Phi)$.

Let us clarify the geometric meaning of the term genus.

We note first that *fitting a pair of holes of a simple normal surface Q_r with a handle does not change the Euler characteristic of the surface.* Indeed, an identification of two contours (divided into the same number of arcs) does not change the number of triangles, while the number of segments and vertices is decreased by the same number (if the boundary of each hole had k sides and k vertices, the total number of 1- and the total number of 0-elements of the triangulation after the matching is decreased by k).

In the same way, closing up one of the holes of the surface Q_r with a Möbius band does not change the Euler characteristic of the surface.

Let p be an arbitrary nonnegative integer and consider the normal simple surface Q_{2p}, i.e., the sphere with p pairs of holes. Let us denote by Φ_p the closed and, as is easily seen, orientable surface obtained by fitting the p pairs of holes with handles of the first kind. By the above remarks, the Euler characteristic of the surface Φ_p is equal to the Euler characteristic of the surface Q_{2p}, i.e., $2 - 2p$. Hence

$$\chi(\Phi_p) = 2 - 2p.$$

Since, on the other hand (see 5.14),

$$\chi(\Phi_p) = 2 - q,$$

it follows that $q = 2p$, i.e., $p = q/2$ is the genus of the surface Φ_p.

Hence

7.12. Let p be an arbitrary nonnegative integer; *fitting each of the p pairs of holes of the normal simple surface Q_{2p} with a handle of the first kind yields a closed orientable surface Φ_p of genus p.*

The surface Φ_p is called the *normal closed orientable surface of genus p or simply "the sphere with p handles".*

Let us now close up all $p + 1$ holes of the surface Q_{p+1} with Möbius bands. The result is a closed nonorientable surface Ψ_p whose Euler characteristic is equal to the Euler characteristic of the surface Q_{p+1}, i.e., $1 - p$.

Euler's formula implies that $p = q - 1$, i.e., that p is the genus of the nonorientable surface Ψ_p.

Hence

7.13. *Closing up all $p + 1$ holes of the surface Q_{p+1} with Möbius bands yields a nonorientable closed surface Ψ_p of genus p.*

It is called the *normal closed nonorientable surface of genus p* (or the *sphere with $p + 1$ cross-caps*).

In view of the above construction it is natural to ask what happens if a pair of holes of the surface Q_r is fitted with a handle of the second kind. The answer to this question is given by the following proposition, which we shall need.

7.14. *Fitting a pair of holes of the surface Q_r with a handle of the second kind is equivalent to fitting each of these holes with a Möbius band* (and therefore leads to a nonorientable surface).

Proof. Fig. 68 shows a pair of holes fitted with a handle of the second kind; the points 1, 2, 3, 4, 5, 6, 7, 8 are identified with the points 1′, 2′, 3′, 4′, 5′, 6′, 7′, 8′, respectively. Let us perform cross cuts along the lines 33′ and 77′ (Fig. 69) and turn the "quadrilateral" (Figs. 69, 70) 42′8′6 through 180° about the axis AA' (Fig. 71).

Matching the elements to be identified (*denoted by identical numbers*), we get Fig. 72.

It now remains merely to close up each of the two holes $3ab3'a'b'$ and $7d'c'7'dc$ as indicated (i.e., to identify 3 with 3′, a with

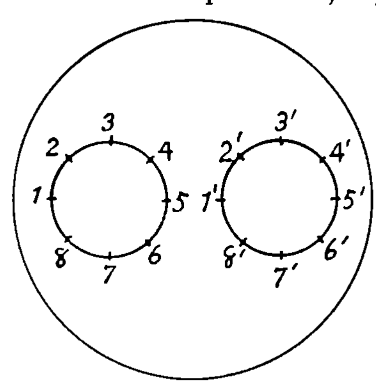


FIG. 68

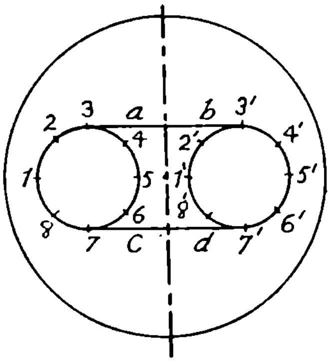


FIG. 69

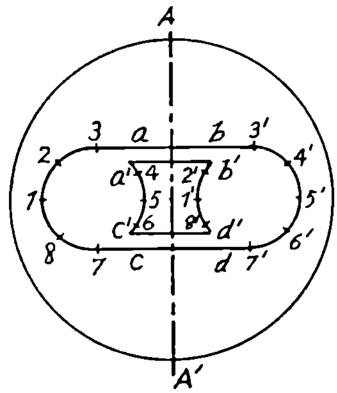


FIG. 70

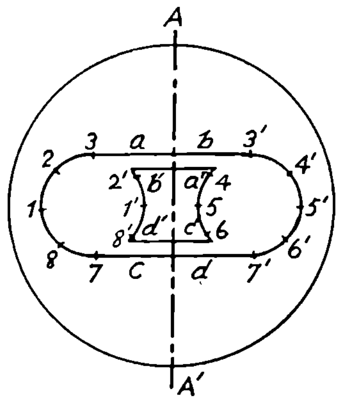


FIG. 71

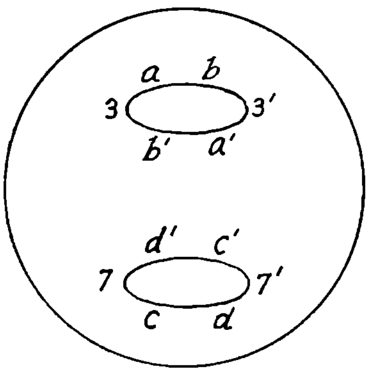


FIG. 72

a', b with b', etc.). These are reductions of the second kind, i.e., each of the holes is closed with a Möbius band.

§7.2. The fundamental theorem of the topology of surfaces. Our purpose will be achieved by proving the following two propositions, which together make up the so called fundamental theorem of the topology of surfaces.

7.2. *Every closed surface* Φ *is homeomorphic to a normal closed orientable surface* Φ_p *of some (integral) genus* p *(if* Φ *is orientable) or to a normal closed nonorientable surface* Ψ_p *of genus* p *(if* Φ *is nonorientable).*

It follows immediately that the genus $p(\Phi)$ of every closed orientable surface Φ is an integer. Therefore the connectivity is an even number.

Theorem 7.2 and the invariance of the Euler characteristic and orientability of the surface imply

7.21. *Two closed surfaces* Φ *and* Φ' *are homeomorphic if, and only if, they are both orientable or both nonorientable and if, in addition, they have either the same Euler characteristic, or the same connectivity, or the same genus.*

Passing to the proof of 7.2, we note first that:

7.211. Let Φ be a closed surface with $q = q(\Phi)$. Every subcomplex of an arbitrary triangulation K of the surface Φ consisting of $q + 1$ mutually disjoint closed curves

$$L_1, \cdots, L_{q+1},$$

separates the triangulation.

Indeed, $L_1 \cup \cdots \cup L_{q+1}$ is a 1-complex of connectivity $q + 1$, whence 7.211 follows.

Let $q' = q'(\Phi)$ be the maximum number such that there exist q' mutually disjoint closed curves $L_1, \cdots, L_{q'}$ on some triangulation K of the surface Φ which form a nonseparating subcomplex $L = L_1 \cup \cdots \cup L_{q'}$ of K. It follows from 7.211 that

$$q'(\Phi) \leq q.$$

Let us perform a cut along all the curves $L_1, \cdots, L_{q'}$. As a result of this cut, the triangulation $K = K_0$ is converted into a triangulation K_1 of some surface $\| K_1 \| = \Phi_1$.

Let us show that K_1 is a simple triangulation. In fact, if there were a nonseparating cut on the triangulation K_1, there would exist a nonseparating cut on some subdivision K'_1 of K_1, consisting entirely of interior elements. Consequently, $q' + 1$ mutually disjoint nonseparating cuts could be performed on the corresponding triangulation K' of the surface Φ. This contradicts the definition of q'.

Hence Φ_1 is a simple surface homeomorphic, by 6.44, to some normal simple surface Q_r.

In order to return to Φ from Φ_1 it is necessary to perform an identification suppressing the cuts made on the surface Φ along the lines $L_1, \cdots, L_{q'}$. At the same time we will perform the corresponding identification on Q_r. As a result, Φ_1 becomes Φ and Q_r is turned into some closed surface S homeomorphic to Φ.

What will S represent? To answer this question, let us consider in detail a cut of Φ along a fixed curve $L = L_i$ and the inverse operation, an identification. The following cases are possible:

1. The cut L is one-sided.

Its suppression is a reduction of the second kind to which, on Q_r, corresponds closing up a hole with a Möbius band. If Φ is an orientable surface, this case cannot arise.

2. The cut L is two-sided; it generates two contours L^+, L^-, to which correspond two holes L'_1, L'_2 of the surface Q_r. Therefore, if every cut L is two-sided, r is an even number $2p$.

Suppression of the cut L of the surface Φ corresponds on Q_r to the reduction of the pair of holes L'_1, L'_2, which is realized either by a handle of the first kind or a handle of the second kind. In the former case we call the cut L a two-sided cut of the first kind and in the latter case a two-sided cut of the second kind.

By 7.14, it is impossible to have a two-sided cut of the second kind on an orientable surface.

Hence, if Φ is an orientable closed surface, r is an even number $2p$, and the surface S, the homeomorph of Φ, is obtained from Q_r by fitting each pair of holes with a handle of the first kind.

In other words:

An integer $p \geq 0$ is defined for every orientable closed surface Φ such that Φ is homeomorphic to a normal orientable surface of genus p ("sphere with p handles") and is consequently itself a surface of genus p: two orientable closed surfaces are homeomorphic if, and only if, they have the same genus or, what comes to the same, the same Euler characteristic.

Let us now pass to the case of a closed nonorientable surface. Cuts of all three types may occur when a closed nonorientable surface Φ is transformed into a simple surface Q_r. Moreover, either at least one one-sided cut, or at least one two-sided cut of the second kind (or both) must occur.

Hence, every closed nonorientable surface is homeomorphic to a closed surface S resulting from some simple surface Q_r by means of the following operations:

a) closing up some of the holes of the surface Q_r with Möbius bands;

b) fitting some pairs of holes of the surface Q_r with handles of the second kind;

c) fitting some pairs of holes of the surface Q_r with handles of the first kind.

We shall show that *it is always possible to restrict oneself to operations of type* a).

First, by virtue of 7.14, every operation of type b) can be replaced by two operations of type a).

We shall now prove

7.22. *The set of two operations:*

1) *closing up a hole with a Möbius band,*

2) *fitting a pair of holes with a handle of the first kind,*

is equivalent to the set of two operations:

1) *closing up a hole with a Möbius band,*

2) *fitting a pair of holes with a handle of the second kind.*

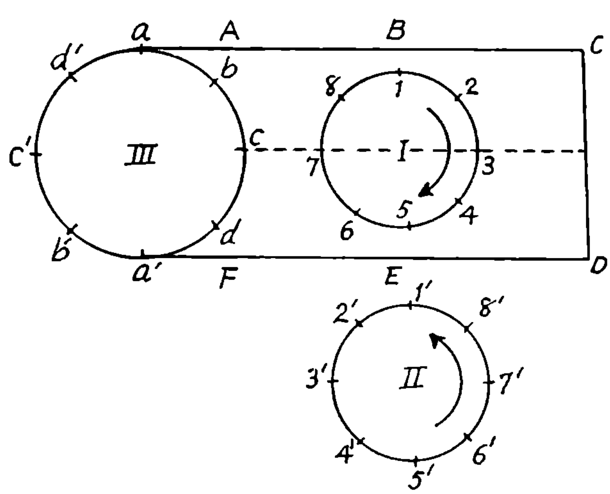


FIG. 73

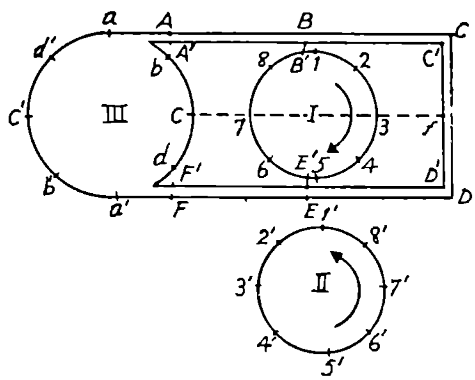


FIG. 74

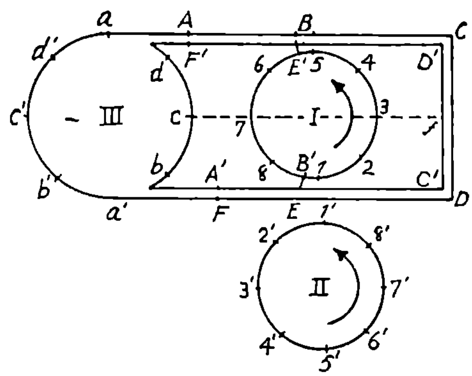


FIG. 75

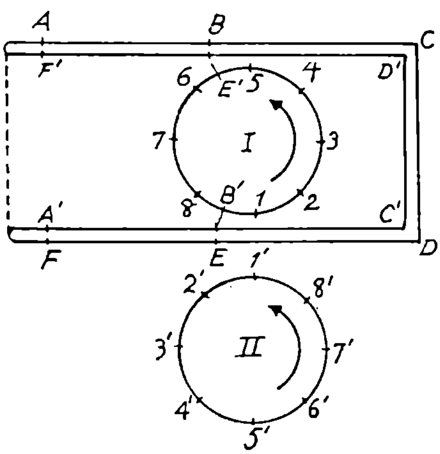


FIG. 76

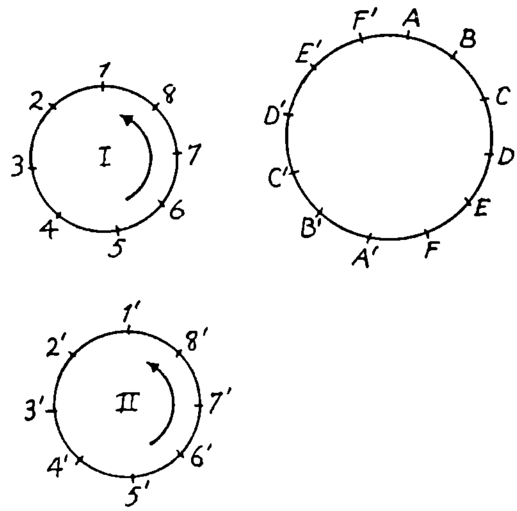


FIG. 77

Indeed, let holes I and II (Fig. 73) be fitted with a handle of the first kind, and hole III be closed with a Möbius band, i.e., its diametrically opposite points identified (a with a', b with b', etc.).

Let us perform a cut along the curve $aABCDEFa'$ (Fig. 73), and turn the figure $A'B'C'D'E'F'dcb$ (Fig. 74) through 180° about the horizontal axis cf (Fig. 75; in Figs. 74 and 75 the axis cf is dotted).

Let us close up the hole III, using the indicated identification (Fig. 76).

Let us add the contour $ABCDEFA'B'C'D'E'F'A$ in the form of a circumference (Fig. 77).

The resulting circular hole is obviously closed up with a Möbius band, and the holes I and II with a handle of the second kind, which was to be proved.

Suppose that a nonorientable surface Φ is obtained from the surface Q_r by a number of operations of types a), b), and c). Since every operation of type b) can be replaced by operations of type a), we may suppose that operations of type a) are present. But in that case every operation of type

c) can be replaced by an operation of type b), so that all are reduced to operations a) and b).

Replacing all operations of type b) with operations of type a), the entire construction of a nonorientable closed surface Φ from a simple surface Q_r is reduced to operations of one type a) alone.

Hence

Every closed nonorientable surface is homeomorphic to a surface obtained from a normal simple surface Q_r (sphere with r holes) by closing up the holes of the surface Q_r with Möbius bands.

This proves Theorem 7.2 in its entirety.

EXAMPLES.

1. For the projective plane Φ we have

$$\chi(\Phi) = 1, \qquad q(\Phi) = 1, \qquad p(\Phi) = 0.$$

The projective plane is obtained by closing up a sphere with one hole with a Möbius band or (what comes to the same) by pasting together a Möbius band and a circle at their boundaries.

2. For a Klein bottle we have

$$\chi(\Phi) = 0, \qquad q(\Phi) = 2, \qquad p(\Phi) = 1.$$

The Klein bottle is obtained if two holes cut in the sphere are closed up with Möbius bands or (what is clearly the same) if two Möbius bands are pasted together along their boundaries.

REMARK 1. From all the above it follows that:

The connectivity of a closed orientable surface Φ is an even number equal to the number of holes in the sphere it is necessary to fit (by pairs) with handles of the first kind in order to obtain a surface homeomorphic to the given surface Φ.

The connnectivity of a closed nonorientable surface Φ is equal to the number of holes in the sphere which it is necessary to close up with Möbius bands in order to obtain a surface homeomorphic to the given surface.

REMARK 2. If the connectivity of a given closed nonorientable surface Φ is an even number $q = 2p$, a surface homeomorphic to Φ can be obtained by taking the simple surface Q_{2p} (the sphere with $2p$ holes) and fitting the p pairs of holes of this surface with handles of the second kind (if $p = 1$, the surface is the Klein bottle).

Part Two

COMPLEXES. COVERINGS. DIMENSION

This part, like the preceding one, consists of three chapters. Chapter IV is a detailed study of the notion of a complex; hence this chapter (and also Chapter VII) contains a development of the combinatorial (and algebraic) apparatus of topology; both chapters are essentially auxiliary in character. Chapter VII can be read immediately after Chapter IV.

Several important topological facts—the invariance of the dimension number of polyhedra, the invariance of interior points, and the fixed point theorem for continuous mappings of a simplex—are proved in Chapter V by elementary means (with the aid of Sperner's lemma). Chapter V may be read after Chapter IV.

Chapter VI is devoted to an introduction to dimension theory and is based on IV; 1, 2.1–2.2, 3, 4.1–4.3, and 5.1–5.3 (and on the portions of Chapter I indicated in the references). Chapter VI should be read after Chapter V.

Chapter IV

COMPLEXES

Introductory section: preliminary remarks on simplexes

In this section we shall recall the definition of an n-simplex (see Appendix 1) and introduce the basic notions associated with this definition.

The reader is advised to omit on a first reading the parts of this section marked with an asterisk. They are not needed until Chapter XIV.

§0.1. Simplexes and their skeletons. Let $e_0, e_1, \cdots, e_n$, $0 \leq n \leq m$, be $n + 1$ linearly independent points in the Euclidean m-space R^m; in virtue of their linear independence, these points determine an n-plane $R^n(e_0, e_1, \cdots, e_n) \subseteq R^m$ and in that plane a system of barycentric coordinates (see Appendix 1, 1.2). Those points of the plane $R^n(e_0, e_1, \cdots, e_n)$ all of whose barycentric coordinates $\mu_0, \mu_1, \cdots, \mu_n$ are positive with respect to the system $(e_0, e_1, \cdots, e_n)$ form, by definition, an *n-simplex $T = (e_0 \cdots e_n)$ with vertices $e_0, \cdots, e_n$.*

The number of vertices of a simplex diminished by 1 is called the *dimension number* or *dimension* of the simplex. The dimension of a simplex is usually denoted by a superscript. Thus, T^n denotes an n-simplex.

A 0-simplex is a point. A 1-simplex with vertices e_0 and e_1 is an open segment (e_0e_1) (a segment without its endpoints). A 2-simplex is a triangle and a 3-simplex, a tetrahedron (also open, i.e., without their boundary points).

DEFINITION 0.11. *The set of all $n + 1$ vertices of an n-simplex is called the skeleton of the simplex.*

Simplexes in R^m correspond (1–1) to their skeletons.

*REMARK. It is sometimes expedient to consider systems of $n + 1$ distinct, but linearly dependent, points in R^m as the skeletons of "degenerate" n-simplexes. In that case it is convenient to identify a degenerate n-simplex with its skeleton.

Hence we arrive at the definition:

Every set of $n + 1$ points of R^m lying in a plane $R^k \subseteq R^m$, $k < n$, is called a degenerate n-simplex of R^m. By the skeleton of a degenerate simplex we shall understand the degenerate simplex itself. (Wherever degenerate simplexes are used, this will be explicitly mentioned. Otherwise, simplex will mean nondegenerate simplex.)*

§0.2. Faces. If the skeleton of a simplex T_1 is a subset of the skeleton of a simplex T_2, T_1 is said to be a *face* of T_2. If $T_1 = (e_0 \cdots e_r)$ is a face of $T_2 = (e_0 \cdots e_re_{r+1} \cdots e_n)$, T_1 consists of all the points of the space R^n

$(e_0, \cdots, e_n)$ whose barycentric coordinates μ_i with respect to $e_0, \cdots, e_n$ are positive for $i \leq r$ and equal to zero for $i > r$.

The number of r-dimensional faces (r-faces) of an n-simplex is equal to the number of combinations $C(r + 1, n + 1)$ of $n + 1$ things taken $r + 1$ at a time. The 0-faces of a simplex are its vertices. A simplex is its own face; the remaining faces are said to be *proper faces* of the simplex.

REMARK 1. It is important to observe that in this book a *face* (unless otherwise noted) of a simplex will always mean either a proper face of the simplex or the simplex itself.

If a simplex T_1 is a *proper* face of a simplex T_2, we shall write $T_1 < T_2$ or $T_2 > T_1$. If T_1 is a face of T_2, we shall write $T_1 \leq T_2$ or $T_2 \geq T_1$.

The simplexes T_1 and T_2 are said to be *incident* if either $T_1 < T_2$ or $T_2 < T_1$.

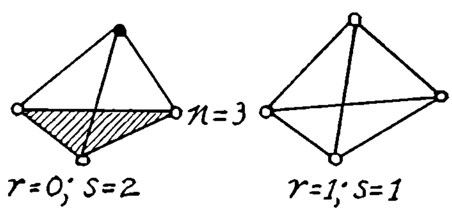


FIG. 78

The faces $T^r{}_1$ and $T^s{}_2$ of a simplex T^n are said to be opposite faces of T^n if every vertex of T^n is a vertex of one of the faces $T^r{}_1$ or $T^s{}_2$ and if these faces have no vertices in common (Fig. 78). In that case

$$r + s = n - 1.$$

It is sometimes convenient to denote the face T^r of a simplex $(e_0 \cdots e_n)$ opposite the face T^s of this simplex by placing the symbol "$\wedge$" over the vertices of the face T^s; e.g., $(e_0 \cdots \hat{e}_i \cdots e_n)$ denotes the face

$$(e_0 \cdots e_{i-1}e_{i+1} \cdots e_n).$$

REMARK 2. If $T^n = (e_0 \cdots e_n)$ is a degenerate simplex (or, what is the same in our terminology, a degenerate skeleton), every simplex (degenerate or not) whose skeleton is a proper subset of the skeleton $(e_0 \cdots e_n)$ is called a proper face of T^n. This definition implies that a degenerate simplex may have nondegenerate faces, e.g., three points A, B, C lying on a straight line form a degenerate triangle which, however, has three nondegenerate sides AB, BC, CA.

§0.3. Combinatorial sum. Let $T_1, T_2, \cdots, T_k$ be any faces of a simplex T. The union of the skeletons of the simplexes $T_1, T_2, \cdots T_k$ is a subset of the skeleton of the simplex T and is therefore the skeleton of some face

$T_0 \leq T$. The simplex T_0 is called the *combinatorial sum* of the simplexes $T_1, T_2, \cdots, T_k$ and is denoted by $(T_1 T_2 \cdots T_k)$.

The combinatorial sum is therefore defined for simplexes which are faces of the same simplex.

Obviously:

1. Every simplex is the combinatorial sum of any two opposite faces of the simplex.

2. Every simplex is the combinatorial sum of all its faces of a given dimension (in particular, of all its vertices).

§0.4. The closure of a simplex $T^n = (e_0 e_1 \cdots e_n)$ in R^m is denoted by $\overline{T}^n = [e_0 \cdots e_n]$ and is referred to as a *closed simplex*. A closed simplex $\overline{T}^n = [e_0 e_1 \cdots e_n]$ consists of all those points of the space $R^n(e_0, e_1, \cdots, e_n)$ whose barycentric coordinates with respect to $e_0, e_1, \cdots, e_n$ are non-negative.

The *point set boundary* or *frontier* $\overline{T}^n \setminus T^n$ of a simplex T^n is the (point set) union of all the proper faces of the simplex T^n and is denoted by $\| \dot{T}^n \|$ (for reasons which will be made clear in the sequel).

§1. Basic definitions

§1.1. Triangulations. Examples: $| T^n |$ and $\dot{T}^n$.

DEFINITION 1.11. A finite set of *mutually disjoint* simplexes situated in some R^n is called a *triangulation* if every face of every simplex of K is also an element of K. The set K may be partially ordered as follows: a simplex $T_1 \in K$ precedes a simplex $T_2 \in K$ ($T_1 < T_2$) if T_1 is a proper face of T_2.

The maximum dimension of the simplexes of K is called the *dimension number* or the *dimension* of K.

REMARK. The naturalness of this definition follows from, e.g., the reasoning and results of Chapter III: a considerable part of topology consists of the study of the so called polyhedra, i.e., the sets $\Phi \subset R^n$ which can be "triangulated" (split up into simplexes which form a triangulation); in other words, sets which are the (point set) *union of the simplexes of some triangulation*. At the same time we shall, of course, study *curved* (*topological*) *polyhedra*: topological images of triangulable sets. Topological polyhedra are a far-reaching multi-dimensional generalization of the closed surfaces of Chapter III. The investigation of the topological properties of curved polyhedra exhausts the study of polyhedral topology, since curved polyhedra are homeomorphic to triangulations, and hence both have the same topological properties. Triangulation is the basic auxiliary technique of polyhedral topology; not only is it one of the most important parts of topology, but it also suggests techniques for studying more general

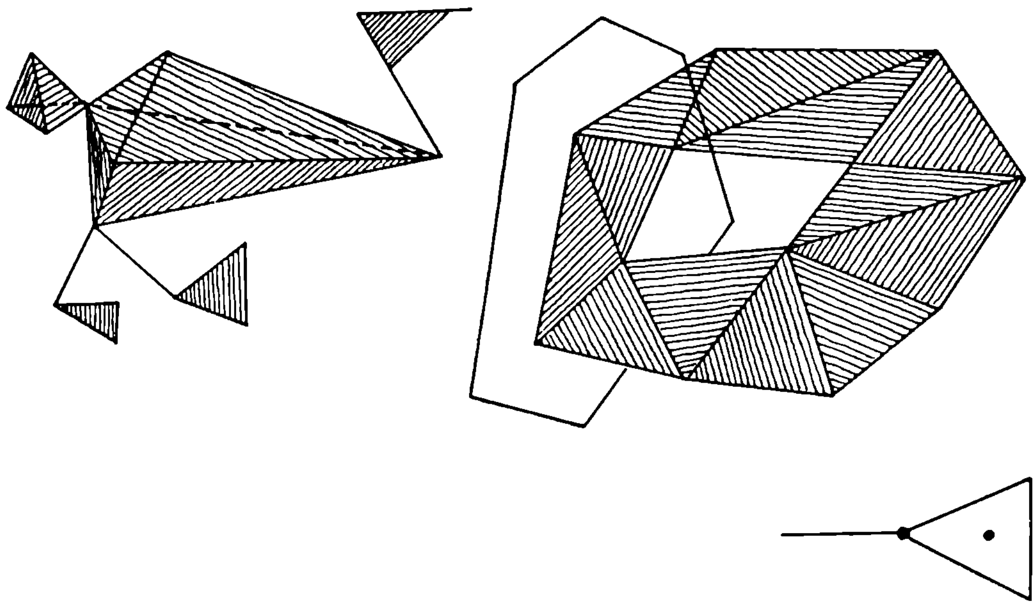


FIG. 79

topological spaces, to begin with, compacta. Hence, the enormous significance of triangulations in modern topology is understandable. Triangulations are not the ultimate objects of study in topology and occupy a subordinate position in this respect, but they and their immediate generalizations, as a technique, are basic to modern topological research. This peculiar position of triangulations, at once subordinate and fundamental, should be clear from Chapter III (and Chapter V).

Examples of triangulations are easily constructed. Several examples of one-, two-, and three-dimensional triangulations are shown in Fig. 79 and also in the figures of Chapter III. The set of all proper faces of a tetrahedron, an octahedron, or an icosahedron is also an example of a two-dimensional triangulation.

We shall give special attention to two elementary examples constantly required in the sequel.

1. The triangulation consisting of an n-simplex T^n and all its faces is denoted by $| T^n |$ and called the *combinatorial closure of the simplex* T^n.

2. The $(n - 1)$-dimensional triangulation consisting of all the proper faces of an n-simplex T^n is denoted by $\dot{T}^n$ and is called the (*combinatorial*) *boundary of the simplex* T^n (as distinguished from the frontier $\| \dot{T}^n \|$ of T^n, which is the point set union of all the proper faces of T^n).

§1.2. Polyhedral complexes. The definition of a *polyhedral complex* is obtained by replacing the word "simplex" in the definition of a triangulation by "convex polyhedral domain" (see Appendix 1), leaving everything

else unchanged. In this connection, convex polyhedral domains (as well as simplexes) are regarded as open, i.e., without frontier points: a square means the interior of a square, a cube, the interior of a cube, etc. This concept, at least in significance, is less important than the notion of a triangulation, but it is so natural a generalization of the latter, that it is impossible to give it up. Besides, sometimes replacing a triangulation of a given polyhedron by a more general polyhedral complex may simplify some construction. Examples of polyhedral complexes are: the complex consisting of all the faces of a cube, or a dodecahedron; the decomposition into quadrilaterals, their sides and vertices, of a surface homeomorphic to a torus shown in Fig. 80, and many others.

If T^n is a convex polyhedral domain, its combinatorial closure $| T^n |$ and its boundary $\dot{T}^n$ are defined exactly as in the case of a simplex (see the preceding article).

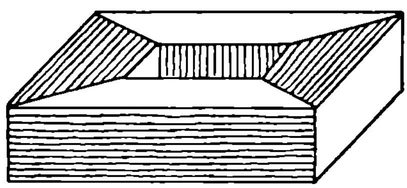


FIG. 80

§1.3. Skeleton complexes. This notion is important in the highest degree, especially for a geometric treatment of the questions of set-theoretic topology; but, in this book, the applications of skeleton complexes to the problems of polyhedral topology are essential. We shall preface the definition of skeleton complexes with the following remarks.

Simplexes, in particular simplexes of a given triangulation K, correspond (1–1) to their skeletons, where a simplex $T_1 \in K$, precedes a simplex T_2 (i.e., is a proper face of T_2) if, and only if, the skeleton of T_1 is a proper subset of the skeleton of T_2.

In many questions relating to triangulations, the point set character of simplexes is unessential. What is significant is the order prevailing in the triangulation as a partially ordered set, and the fact that each simplex is assigned a dimension. In all *purely combinatorial questions* it is natural and convenient to replace the given triangulation by the set of skeletons corresponding to it, i.e., the set of skeletons of the simplexes of the given triangulation, where the dimension assigned to each skeleton is the number of vertices making up the skeleton less 1. Hence we arrive at the so called skeleton complexes whose definition will now be given in a very general form.

DEFINITION 1.31. *Let M be a set of elements, called vertices. Let certain*

nonempty finite subsets of M be singled out and called skeletons. The number of vertices in the skeleton less 1 *is called the dimension of the skeleton. The given set of skeletons is partially ordered by means of the natural order* (I, 6.2, Example 4; i.e., T_1 *precedes* T_2 *if* $T_1 \subset T_2$); *it is called a skeleton complex.*

Hence *a skeleton complex is an arbitrary collection of certain finite subsets of a set M.*

A skeleton complex is said to be *unrestricted* if every nonempty subset of a skeleton of K is also a skeleton (i e., an element of K).

REMARK. The skeleton complex of a given triangulation (i.e., the skeleton complex of all the elements of this triangulation) is a finite unrestricted skeleton complex. Finite, and especially unrestricted, complexes are the most important of the skeleton complexes. However, restricted complexes are also often used.

DEFINITION 1.32. Let K be a skeleton complex, and let $| K |$ be the skeleton complex such that every nonempty subset of every skeleton of K is a skeleton of $| K |$. The skeleton complex $| K |$ is unrestricted. We shall call it the *combinatorial closure of the complex K.*

Obviously, K is a subcomplex of $| K |$. Hence

1.33. Every skeleton complex is a subcomplex of an unrestricted skeleton complex.

§1.4. General definition of a simplicial complex. We shall begin with the following example. Let us consider a tetrahedron T inscribed in a solid sphere. Its vertices lie on the surface of the sphere and determine, on the one hand, the faces and edges of the tetrahedron and, on the other hand, the spherical triangles and their sides obtained by a central projection of these faces and edges. Hence the same skeleton complex corresponds, on the one hand, to an ordinary triangulation T and, on the other hand, to a "curvilinear triangulation of the sphere", whose elements have the same skeletons as those of the triangulation T. Both kinds of elements are "simplexes", the former the usual, the latter the spherical, defined by the given skeletons; it is natural to treat both sets of "simplexes" as "simplicial complexes". Thus we arrive at the following very general and final definition.

DEFINITION 1.41. Let K be an unrestricted skeleton complex; let each skeleton $\{e_0, \cdots, e_r\}$ of K be made to correspond to a unique object $(e_0 \cdots e_r)$ called an *abstract simplex* with vertices $e_0 \cdots, e_r$; in this connection it is required that the correspondence be (1–1) (distinct skeletons correspond to distinct simplexes) and that it assign to each 0-dimensional skeleton, i.e., to each vertex e, as its simplex the vertex e itself. The skeleton $\{e_0, \cdots, e_r\}$ is referred to as the skeleton of the simplex $(e_0 \cdots e_r)$, and

the dimension r of the skeleton $\{e_0, \cdots, e_r\}$ is also the dimension of the simplex $(e_0 \cdots e_r)$.

A simplex $(e'_0 \cdots e'_r)$ precedes a simplex $(e_0 \cdots e_r)$ or is a proper face of the simplex $(e_0 \cdots e_r)$ if the skeleton $\{e'_0, \cdots, e'_r\}$ is a proper subset of the skeleton $\{e_0, \cdots, e_r\}$.

The set of all abstract simplexes constructed in this way is called an *unrestricted simplicial complex* and the original skeleton complex K is called the *skeleton complex of the simplicial complex.*

In accordance with the above, every unrestricted simplicial complex is a partially ordered set.

This definition implies that all unrestricted skeleton complexes, as well as all triangulations, are special cases of unrestricted simplicial complexes. [In the first case an abstract simplex $(e_0 \cdots e_r)$ with skeleton $\{e_0, \cdots, e_r\}$ is itself that skeleton, in the second case it is the usual simplex with vertices $e_0, \cdots, e_r$.] A curved triangulation is also an unrestricted simplicial complex (see §6), whose two-dimensional case was discussed in Chapter III.

DEFINITION 1.42. Every subset ("subcomplex") K_0 of an unrestricted simplicial complex K is called simply a simplicial complex. The elements of the complex K_0 have the same skeletons, the same dimensions and the same order as in K. The combinatorial closure $| K_0 |$ of a complex K_0 (in K) is, by definition, the unrestricted subcomplex of K, whose skeleton complex is the combinatorial closure of the skeleton complex K_0.

DEFINITION 1.43. If the elements of a simplicial complex K have a maximum dimension, this maximum is called the *dimension (number) of the simplicial complex* K; if a maximum does not exist, the complex K is said to be *infinite dimensional.*

§1.5. Examples of simplicial complexes. 1. Let K be a triangulation, and K_0 a subcomplex of K; K_0 is a simplicial complex.

2. Let Γ be an open subset of R^n or the n-sphere S^n. A finite set

$$\{e_0, \cdots, e_r\}$$

contained in Γ is called a skeleton if its closed convex hull is in Γ (if $\Gamma \subset S^n$, it is required, in addition, that the set $\{e_0, \cdots, e_r\}$ be contained in some hemisphere of S^n and that closed convex hull be understood in the sense of the spherical metric in S^n). The resulting unrestricted skeleton complex is denoted by $K(\Gamma)$. The subcomplex of $K(\Gamma)$ consisting of all skeletons whose diameter is less than some definite positive number ϵ, is an unrestricted skeleton complex denoted by $K(\Gamma, \epsilon)$.

Let us consider the simplicial complex having $K(\Gamma)$, $K(\Gamma, \epsilon)$, respectively, as its skeleton complex, with its simplexes defined as follows: if the points $e_0, \cdots, e_r$ of a skeleton are linearly independent in R^n (in S^n), the

ordinary (or spherical) simplex with vertices $e_0, \cdots, e_r$ is called a simplex $(e_0 \cdots e_r)$.

But if the points $e_0, \cdots, e_r$ are linearly dependent (which, in particular, will be true if $r > n$), then set

$$(e_0 \cdots e_r) = \{e_0, \cdots, e_r\},$$

i.e., identify the simplex $(e_0 \cdots e_r)$ with its skeleton. The resulting simplicial complex is at times also denoted by $K(\Gamma)$ or $K(\Gamma, \epsilon)$, as the case may be.

3. Let R be a metric space. Denote by $K(R)$ the unrestricted skeleton complex obtained by letting every finite subset of R be a skeleton; $K(R)$ contains a subcomplex $K(R, \epsilon)$ consisting of all the skeletons of the complex $K(R)$ whose diameter is $<\epsilon$.

REMARK. The case of R a compactum is especially important.

If R is a subset of R^n, the complexes $K(R)$ and $K(R, \epsilon)$ can be replaced by simplicial complexes with the same skeletons but with simplexes defined as at the end of Example 2.

4. Let Γ be an open subset of a compactum Φ. Let us call every finite subset of Φ which has a nonempty intersection with Γ a skeleton. The resulting skeleton complex is denoted by $K(\Phi, \Gamma)$; if it were required that every skeleton have a diameter $<\epsilon$, the result would be a skeleton complex $K(\Phi, \Gamma, \epsilon) \subset K(\Phi, \Gamma)$.

Other important examples of skeleton complexes are discussed in §2.

§1.6. Simplicial mappings and isomorphisms of skeleton and simplicial complexes.

DEFINITION 1.61. Let us assign to every vertex e_β of an unrestricted skeleton complex K_β a vertex $e_\alpha = S_\alpha^\beta e_\alpha$ of a skeleton complex K_α in such a way that the image of every skeleton $\{e_{\beta_0}, \cdots, e_{\beta_r}\}$ of K_β is a skeleton of the complex K_α (distinct vertices of K_β may correspond to a single vertex of K_α). In virtue of the condition imposed, the mapping S_α^β of the set of vertices of K_β into the set of vertices of K_α induces a mapping of the complex K_β into the complex K_α, referred to as a *simplicial mapping of the unrestricted skeleton complex K_β into the skeleton complex K_α*, also denoted by S_α^β.

If K_β and K_α are the unrestricted skeleton complexes of the unrestricted simplicial complexes K'_β and K'_α, respectively, then an assignment to each simplex $(e_{\beta_0} \cdots e_{\beta_r}) \in K'_\beta$ of the simplex of the complex K'_α whose vertices are $S_\alpha^\beta e_{\beta_0}, \cdots S_\alpha^\beta e_{\beta_r}$ yields, by definition, a simplicial mapping S_α^β of the simplicial complex K'_β into the simplicial complex K'_α (induced by the identically designated simplicial mapping of the skeleton complex K'_β into the skeleton complex K'_α).

DEFINITION 1.62. Let K_β be a restricted simplicial complex. A simplicial

mapping of K_β is any simplicial mapping of the combinatorial closure $|K_\beta|$ of K_β restricted to K_β.

DEFINITION 1.63. *A (1–1) simplicial mapping of an unrestricted simplicial complex K_β onto* (i.e., every skeleton of K_α is the image of at least one skeleton of K_β) *an unrestricted simplicial complex K_α is called an isomorphic mapping or an isomorphism. Two unrestricted simplicial complexes are said to be isomorphic or to have the same combinatorial type if there is an isomorphic mapping of one onto the other.*

Two unrestricted simplicial complexes are isomorphic if, and only if, their skeleton complexes are isomorphic (in particular, every unrestricted simplicial complex is isomorphic to its skeleton complex). Hence the combinatorial type of an unrestricted simplicial complex is uniquely determined by its skeleton complex.

REMARK ON ISOMORPHISM. A simplicial mapping S_α^β of an unrestricted simplicial complex K_β onto an unrestricted simplicial complex K_σ is an isomorphism if, and only if, every vertex of K_α is the image of exactly one vertex of K_β.

EXAMPLES OF ISOMORPHIC SIMPLICIAL COMPLEXES.

1. The triangulation consisting of all the proper faces of a convex polyhedral domain with triangular faces (e.g., a tetrahedron, an octahedron, an icosahedron, etc.) is isomorphic to the "curved triangulation" of the sphere obtained by projecting the polyhedral domain onto a sphere in its interior.

2. Let K be a triangulation. Let us construct the simplicial complex K' with the same skeleton complex as that of K but with simplexes which are defined as the closures of the simplexes of K.

The complex K' is isomorphic to the complex K.

REMARK ON IDENTIFICATIONS. Let K_β and K_α be two simplicial complexes and S_α^β a simplicial mapping of K_β onto K_α:

$$S_\alpha^\beta K_\beta = K_\alpha .$$

To every vertex $e_{\alpha i} \in K_\alpha$ corresponds a class $\mathfrak{e}_i$ of vertices $e_{\beta j}$ of K_β such that

$$S_\alpha^\beta\, e_{\beta j} = e_{\alpha i} .$$

Hence the simplicial mapping S_α^β induces a decomposition of the set of all vertices $e_{\beta j}$ of the complex K_β into classes. We note that the vertices $e_{\alpha 0}, \cdots, e_{\alpha r} \in K_\alpha$ form a skeleton in K_α if, and only if, the corresponding classes $\mathfrak{e}_0, \cdots, \mathfrak{e}_r$ contain vertices $e_{\beta j} \in \mathfrak{e}_j$, $j = 0, 1, \cdots, r$, which form a skeleton in K_β.

Hence the simplicial mapping S_α^β is equivalent to an *identification* (see I, 5.1 and III, 3.1) of all the vertices of K_β which are in the same class $\mathfrak{e}_j$.

Conversely, let K be a finite unrestricted simplicial complex and suppose that its vertices are divided into classes $\mathfrak{e}_i$. A new simplicial complex K_α may be defined as follows: K_α is a skeleton complex whose vertices are the classes $\mathfrak{e}_i$ of vertices of K_β. The classes $\mathfrak{e}_0, \cdots, \mathfrak{e}_r$ form a skeleton if it is possible to choose vertices $e_{\beta i} \in \mathfrak{e}_i$, $i = 0, 1, \cdots, r$, which form a skeleton in K_β. It is obvious that the result is an unrestricted skeleton complex.

The simplicial mapping S_α^β of K_β onto K_α is defined in the natural way, i.e., each vertex $e_{\beta i} \in K_\beta$ is mapped into the class $\mathfrak{e}_i$ containing it.

Let us suppose that a simplicial mapping of K_β onto K_α satisfies the following condition: S_α^β maps no pair of vertices contained in the same skeleton of the complex K_β into the same vertex of the complex K_α. Such mappings are called *identifications*; an identification maps every element of the complex K_β onto an element of the same dimension of K_α and hence two elements of different dimensions of K_β cannot be mapped on the same element of K_α; but two elements of K_β having the same dimension can be identified, i.e., they can be mapped onto the same element of K_α.

The identifications considered in Chapter III are a special case of this general concept.

§1.7. Definition of an abstract complex. Considering the various definitions of complexes given in this section, it may be noted that *they are all special cases of a partially ordered set* Θ *to each of whose elements* θ *is assigned a nonnegative integer* $d\theta$, *the dimension of the element* θ, *such that* $\theta_1 < \theta_2$ *implies that* $d\theta_1 < d\theta_2$.

We shall take this, for the time being, as the definition of an abstract complex. The abstract complexes considered in Chapter VII will be of a more restricted nature.

A similarity mapping of one complex onto another which preserves dimension is called an isomorphic mapping of the two complexes.

This definition of isomorphism in the case of unrestricted simplicial complexes coincides with that which was given in 1.6.

The notions of abstract complex and isomorphism introduced above enable us to give the definition of an unrestricted simplicial complex the following irreproachably simple and transparent form:

An unrestricted simplicial complex is an abstract complex isomorphic to some unrestricted skeleton complex. Simplicial complexes (not necessarily unrestricted) *are arbitrary subcomplexes of unrestricted simplicial complexes.* (Every subset $K_0 \subseteq K$, whose elements have the same dimensions and the same order as in K, is a subcomplex of K.)

Remark. The notion of isomorphism as applied to restricted simplicial complexes will not interest us: in our exposition every restricted simplicial

complex will always appear as a subcomplex of some perfectly definite unrestricted complex K and we shall consider only those isomorphic mappings of the complex K_0 which are induced by isomorphisms of the unrestricted complex K.

§1.8. Closed and open subcomplexes of a complex K. Combinatorial closures and stars. (In this article complexes are to be taken in the abstract sense as defined in the preceding article.)

A subcomplex K_0 of a complex K is said to be a *closed* (*open*) subcomplex of K if every element of K which precedes (i.e., is less than) some element of K_0 [which follows (i.e., is greater than) some element of K_0] is itself an element of K_0.

The proof of the following propositions may be left to the reader.

1.81. If K_0 is closed (open) in K, $K \setminus K_0$ is open (closed) in K.

1.82. If K_0 is an arbitrary, and K_1 a closed (open), subcomplex of a complex K, $K_0 \cap K_1$ is a closed (open) subcomplex of the complex K_0.

1.83. Every closed subcomplex of an unrestricted simplicial complex is an unrestricted simplicial complex.

DEFINITION 1.84. If K_0 is any subcomplex of a complex K, the complex consisting of all the elements of K_0 and of all the elements of K less than at least one element of K_0 (greater than at least one element of K_0) is called the combinatorial closure $| K_0 |$ (star $O_K K_0$) of the subcomplex K_0 in the complex K.

1.85. For arbitrary $K_0 \subseteq K$, the complex $| K_0 |$ is closed, and the complex $O_K K_0$ is open, in K.

REMARK. In the sequel we shall consider only stars $O_K T$ of individual elements of a complex K, i.e., complexes $O_K K_0$ in the case that K_0 consists of a single element $T \in K$. In this connection, the element T is called the center of the star $O_K T$. Some examples of stars were given in III, Figs. 20–25.

DEFINITION 1.86. The complex

$$B_K T = | O_K T | \setminus O_K T \tag{1.86}$$

of all the elements not contained in the star $O_K T$ and less than at least one element of $O_K T$ is called the *outer boundary* of the star $O_K T$.

Since $O_K T$ is open in K, and hence in $| O_K T |$, $B_K T$, by (1.86) and 1.81, is closed in $| O_K T |$ and consequently in K. Thus

1.87. $B_K T$ is a closed subcomplex of K; if K is an unrestricted simplicial complex, $B_K T$ is an unrestricted simplicial complex.

DEFINITION 1.88. (This definition is not needed until Chapter XIII.) Let K be a finite simplicial complex and let $T^p \in K$. Consider the complex $F_K T^p \subset K$ consisting of all the simplexes $T_i^r \in K$ satisfying the condition:

there exists a simplex $T_i^{p+r+1} \in O_K T^p$ of which T_i^r is the face opposite the face $T^p < T_i^{p+r+1}$. The complex $F_K T^p$ is called the *zone* of the star $O_K T^p$ (in K).

STARS IN SIMPLICIAL COMPLEXES. Let $O_K T_1, O_K T_2, \cdots, O_K T_r$ be stars of a simplicial complex K. In accordance with the definition of a star, the intersection $O_K T_1 \cap \cdots \cap O_K T_r$ consists of all the simplexes $T \in K$ which satisfy all the conditions

$$T_1 \leq T, T_2 \leq T, \cdots, T_r \leq T.$$

EXAMPLE. Let the complex K consist of the ten triangles and of the two vertices p_1 and p_2 shown in Fig. 81. The intersection of the stars $O_K p_1$ and $O_K p_2$ consists of the two hatched triangles.

THEOREM 1.89. *The intersection of stars $O_K T_1, O_K T_2, \cdots, O_K T_r$ of an unrestricted simplicial complex K is nonempty if, and only if, there is a simplex in K having all the simplexes $T_1, T_2, \cdots, T_r$ as faces. If this is the case, denoting by T_0 the combinatorial sum of the simplexes $T_1, \cdots, T_r$, we have*

$$O_K T_1 \cap O_K T_2 \cap \cdots \cap O_K T_r = O_K T_0.$$

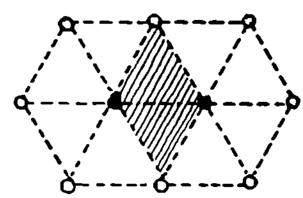


FIG. 81

Proof. Since K is unrestricted, it follows that if the simplexes $T_1, \cdots, T_r$ of K are faces of the same simplex T of K, then the combinatorial sum

$$T_0 = (T_1 T_2 \cdots T_r)$$

of all these simplexes is itself a simplex of K. Obviously, every simplex of K having $T_1, \cdots, T_r$ as faces also has T_0 as a face.

Suppose that a simplex T is in the intersection of the stars $O_K T_1, \cdots, O_K T_r$ of K.

Then the simplexes $T_1, \cdots, T_r$ are faces of T, which consequently also has $T_0 = (T_1 T_2 \cdots T_r)$ as a face, and therefore is an element of the star $O_K T_0$.

Conversely, every simplex $T \in O_K T_0$ has T_0 as a face and consequently has $T_1, \cdots, T_r$ as faces, i.e., is in the intersection of all the stars $O_K T_1, \cdots, O_K T_r$. This proves the theorem.

COROLLARY. In particular, the stars $O_K e_0, O_K e_1, \cdots, O_K e_r$ of vertices $e_0, \cdots, e_r$ of a complex K have a nonempty intersection if, and only if, K contains a simplex $T_0 = (e_0 \cdots e_r)$.

§1.9. Theorem on imbedding in R^{2n+1}.

THEOREM 1.9. *Every finite unrestricted n-dimensional skeleton complex K (hence also every finite unrestricted simplicial n-complex) is isomorphic to*

some triangulation K' situated in Euclidean $(2n + 1)$-space. (The extraordinarily simple proof of Theorem 1.9 given below was communicated to me by L. S. Pontryagin.)

CONSTRUCTION OF THE TRIANGULATION K'. Let $e_1, \cdots, e_s$ be the vertices of the complex K. Take points $e'_1, \cdots, e'_s$ in general position in R^{2n+1} (i.e., such that for $k \leq 2n + 2$ every k of the points $e'_1, \cdots, e'_s$ are linearly independent; see Appendix 1). Put each skeleton $T = \{e_{i_0}, \cdots, e_{i_r}\} \in K$ in correspondence with the simplex $T' = (e'_{i_0} \cdots e'_{i_r}) \subset R^{2n+1}$; this simplex exists since, by virtue of the general position of the points $e'_1, \cdots, e'_s$ in R^{2n+1} and the inequality $r \leq n$, the points $e'_{i_0}, \cdots, e'_{i_r}$ are linearly independent. The simplexes T'_i obviously form a complex K' isomorphic to K. We shall prove that K' is a triangulation. To do this it suffices to prove that every two simplexes $T'_i \in K'$, $T'_j \in K'$ are disjoint. Let the vertices of the simplex T'_i be $e'_{i_0}, \cdots, e'_{i_p}$ and let the vertices of the simplex T'_j be $e'_{j_0}, \cdots, e'_{j_q}$, where some of the vertices may be common to T'_i and T'_j. Let $e'_{k_0}, \cdots, e'_{k_r}$ be all the points which are vertices of at least one of the simplexes T'_i, T'_j. The number $r + 1$ of these points satisfies the inequality

$$r + 1 \leq (p + 1) + (q + 1) \leq (n + 1) + (n + 1) = 2n + 2,$$

since the dimensions of the simplexes T'_i and T'_j do not exceed n. In virtue of the general position of the points $e'_1, \cdots, e'_s$ in R^{2n+1}, the points $e'_{k_0}, \cdots, e'_{k_r}$ are the vertices of some nondegenerate simplex T_0 of dimension $\leq 2n + 1$. The simplexes T'_i and T'_j are faces of the simplex T_0 and hence are disjoint if they are distinct.

1.9 implies

1.91. *Every n-dimensional finite skeleton complex K is isomorphic to a subcomplex of some n-dimensional triangulation whose simplexes are situated in R^{2n+1}.*

Indeed, the combinatorial closure $| K |$ is an unrestricted n-dimensional skeleton complex isomorphic, by 1.9, to a triangulation Q whose simplexes are in R^{2n+1}. The isomorphism between $| K |$ and Q maps the complex $K \subseteq | K |$ onto some subcomplex of Q. This proves 1.91.

§2. Some notable skeleton complexes

§2.1. The nerve of a finite system of sets. This notion will have numerous and very essential applications in this book.

Let

$$\alpha = \{A_1, \cdots, A_s\} \tag{2.1}$$

be a finite system of sets. To each of the sets A_i assign a *vertex* a_i. These vertices (elements of a perfectly arbitrary set; see Def. 1.31)

$$a_1, \cdots, a_s$$

will also be the vertices of a skeleton complex K_α, which we shall now define and call the nerve of the system α. We shall call *a given collection of vertices*

$$a_{i_0}, \cdots, a_{i_r}$$

a skeleton of the complex K_α if, and only if, the sets

$$A_{i_0}, \cdots, A_{i_r}$$

have a nonempty intersection.

REMARK 1. This definition does not exclude the case that several different elements of the system α may coincide as sets [i.e., differ only in their indices in (2.1); see I, 1.3].

REMARK 2. The definition of nerve immediately implies that every nonempty subset of any skeleton of a nerve is also a skeleton of the nerve. Thus:

2.11. *The nerve of every finite system of sets is an unrestricted simplicial complex.*

It follows immediately from the definition of the order of a system of sets (I, 1.3) and the definition of a nerve that:

2.12. *The dimension of the nerve of a system of sets is one less than the order of the system.*

REMARK 3. The nature of the vertices a_i is completely immaterial: any elements which can be put in (1–1) correspondence with the elements of the system α can be taken as the vertices a_i; from the logical point of view it is easiest of all to take as the vertices a_i the elements A_i themselves. Then the nerve of the given system of sets would be uniquely defined and independent of the choice of the vertices.

However, we shall see in the sequel that arbitrariness in the choice of the vertices and a certain indefiniteness arising from it, however unessential, is convenient in practice.

We shall therefore regard the nerve of a system of sets as defined only up to an isomorphism, i.e., we shall call *every complex isomorphic to the nerve of a given system of sets also the nerve of this system of sets.*

Passing to examples of nerves, we note first that the Corollary to Theorem 1.89 can be formulated as follows:

2.13. *The stars of the vertices of a finite unrestricted simplicial complex K form a system of subcomplexes of K having the complex K as its nerve.*

It follows from 2.11 and 2.13 that *every finite unrestricted simplicial complex is the nerve of a finite system of sets and conversely.* Therefore finite unrestricted simplicial complexes are sometimes referred to as simple nerves.

EXAMPLES OF NERVES. 1. Let us consider the system of nine closed squares shown in Fig. 82. The nerve of this system is a 3-complex shown by dotted lines in the same figure.

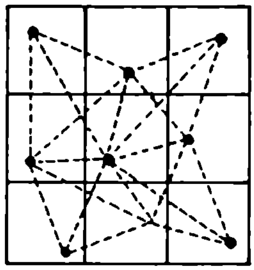


FIG. 82

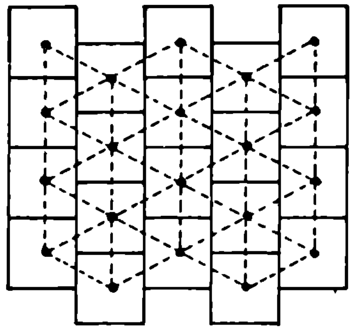


FIG. 83

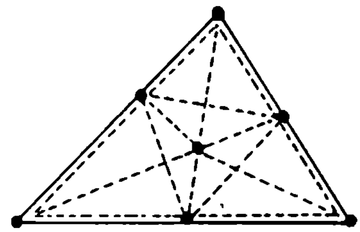


FIG. 84

2. A second example is obtained by taking as the system α the twenty closed squares shown in Fig. 83. The nerve of this system is also shown by dotted lines in the figure.

3. The system α consists of the six closed faces of a cube; its nerve is the complex consisting of all the 2-faces, edges, and vertices of an octahedron.

4. The system α consists of seven closed sets: a closed triangle, its closed sides, and its vertices. The nerve of this system is the 3-complex shown in Fig. 84.

REMARK 4. *The nerve constructed near a given system of point sets.* Let the elements $A_1, \cdots, A_s$ of a system of sets

$$\alpha = \{A_1, \cdots, A_s\}$$

be subsets of a given R^n; let $\epsilon > 0$. Let us take as the vertices $a_1, \cdots, a_s$ of the nerve K_α of α any points $a_1, \cdots, a_s$ of R^n *which are distinct and satisfy the condition*

$$\rho(a_i, A_i) < \epsilon, \qquad i = 1, 2, \cdots, s.$$

A nerve K_α of the system α with vertices $a_1, \cdots, a_s$ satisfying these conditions is called a *nerve contained in an ϵ-neighborhood of α.*

If n is not less than the dimension m of the nerve K_α, let us choose the points $a_1, \cdots, a_s$ in general positon. Then all the skeletons of the nerve are sets of linearly independent points. Hence simplexes of R^n, defined by the corresponding skeletons of the nerve, can be taken as the simplexes of the nerve K_α. *If, in addition, $n \geq 2m + 1$, the resulting nerve K_α will not only be contained in an ϵ-neighborhood of α but will be a triangulation as well.* Hence

2.14. *Let α be a finite system of sets contained in R^n, and let $\epsilon > 0$. It is possible to construct, in every space $R^m \supseteq R^n$ of sufficiently high dimension, a triangulation K_α which is a nerve of α contained in an ϵ-neighborhood of α.*

The nerve of the system of closed squares shown in Fig. 83 is a nerve contained in an ϵ-neighborhood of this system for arbitrary ϵ.

§2.2. Barycentric derivation and barycentric subdivisions. Let Θ be a finite partially ordered set; we shall call its elements p vertices. Those subsets of Θ which are simply ordered (with respect to the order obtaining in Θ), we shall call skeletons.

The resulting (obviously unrestricted) skeleton complex is denoted by $B(\Theta)$ and is called the *barycentric derived* of the partially ordered set Θ.

If Θ, in particular, is a polyhedral complex K or a subcomplex K of a polyhedral complex (see 1.2; the reader may restrict himself to the case of K a triangulation or a subcomplex of a triangulation), it is possible to construct a triangulation K_1 isomorphic to the complex $B(K)$ in a particularly intuitive way. This is done as follows.

Let

$$T_1, \cdots, T_s$$

be all the polyhedral domains of K. All these polyhedral domains are contained in some R^n. Let us choose in each T_i a point e_i called the center of the polyhedral domain; it is customary to choose the centroid of T_i as the point e_i. The vertices of the complex K_1 are the points e_i. A number of vertices e_i form a skeleton if the set of polyhedral domains T_i corresponding to them is simply ordered (with respect to the geometric order, i.e., $T_j < T_i$ if T_j is a proper face of T_i). This means that the given vertices can be written in a sequence

$$e_{i_0}, e_{i_1}, \cdots, e_{i_r}$$

such that

$$T_{i_0} > T_{i_1} > \cdots > T_{i_r} \tag{2.20}$$

(i.e., each of the polyhedral domains $T_{i_0}, \cdots, T_{i_r}$, except T_{i_0}, is a proper face of the polyhedron preceding it).

From (2.20) and the fact that $e_i \in T_i$ it follows that the points $e_{i_0}, \cdots, e_{i_r}$ are linearly independent in R^n, so that they form a simplex $(e_{i_0} \cdots e_{i_r}) \subset R^n$.

Hence *all the skeletons* $\{e_{i_0}, \cdots, e_{i_r}\}$ *are the skeletons of simplexes* $(e_{i_0} \cdots e_{i_r}) \subset R^n$. *These simplexes* $(e_{i_0} \cdots e_{i_r})$ *are also, by definition, the elements of the complex* K_1.

The complex K_1 is isomorphic to the barycentric derived of the arbitrary complex K, by construction: the isomorphism is realized by assigning the simplex $(e_{i_0} \cdots e_{i_r})$ of K_1 to the skeleton $\{T_{i_0} > \cdots > T_{i_r}\}$ of the complex $B(K)$. In consequence, the complex K_1 is called the *geometric realization* of the barycentric derived $B(K)$.

REMARK 1. If $(e_{i_0} \cdots e_{i_r})$ is a simplex of K_1 and

$$T_{i_0} > \cdots > T_{i_r},$$

the vertex e_{i_0} is called the first, and the vertex e_{i_r} the last, vertex of the simplex $(e_{i_0} \cdots e_{i_r})$.

THEOREM 2.21. *The complex K_1 is a triangulation.*

Proof. K_1 is a finite complex, by definition. Since K_1 and $B(K)$ are isomorphic and $B(K)$ is an unrestricted complex, K_1 is also an unrestricted complex.

It remains to be proved that every two distinct simplexes of K_1 are disjoint.

The assertion is obvious for the 0-simplexes of K_1. Before going on to the general case, let us note the following:

Since two distinct elements T_i and T_j of K are disjoint, two simplexes $(e_{i_0} \cdots e_{i_\mu})$, $(e_{j_0} \cdots e_{j_\nu}) \in K_1$ are known to be disjoint if their first vertices e_{i_0} and e_{j_0} are distinct.

Let us now assume that two simplexes of K_1 are disjoint if their dimensions are less than or equal to r. This is true for $r = 0$. Let us consider two simplexes $(e_{i_0} \cdots e_{i_\mu})$, $(e_{j_0} \cdots e_{j_\nu}) \in K_1$ whose dimensions do not exceed $r + 1$. If they are not disjoint, the first vertex of both is $e_{i_0} = e_{j_0}$. But then the simplexes $(e_{i_0} e_{i_1} \cdots e_{i_\mu})$ and $(e_{j_0} e_{j_1} \cdots e_{j_\nu})$ are the projections from the point $e_{i_0} = e_{j_0}$ of the simplexes $(e_{i_1} \cdots e_{i_\mu})$ and $(e_{j_1} \cdots e_{j_\nu})$ and consequently intersect only if $(e_{i_1} \cdots e_{i_\mu})$ and $(e_{j_1} \cdots e_{j_\nu})$ intersect. But the dimensions of $(e_{i_1} \cdots e_{i_\mu})$ and $(e_{j_1} \cdots e_{j_\nu})$ do not exceed r. Therefore, if these simplexes intersect,

$$\mu = \nu, \qquad e_{i_1} = e_{j_1}, \cdots, e_{i_\mu} = e_{j_\nu}.$$

But then $(e_{i_0} \cdots e_{i_\mu})$ and $(e_{j_0} \cdots e_{j_\nu})$ coincide, which was to be proved.

2.210. *Every simplex $T^r{}_1$ of K_1 is wholly contained in some element T of K* (i.e., is a subset of the polyhedral domain T).

Indeed, of the vertices $e_{i_0}, \cdots, e_{i_r}$ of the simplex $T^r{}_1$, the first vertex e_{i_0} is contained in the element $T_{i_0} \in K$ and all the rest in its faces. Therefore the whole simplex $T^r{}_1 = (e_{i_0} \cdots e_{i_r})$ is contained in T_{i_0}.

Now let K be a polyhedral complex (until now K could be an *arbitrary subcomplex* of a polyhedral complex). Then the triangulation K_1 is called the *barycentric subdivision* of K and Theorem 2.210 is completed in an essential way by the proposition:

2.22. *Every element T^k of the complex K is the union of the simplexes of the barycentric subdivision K_1 contained in T^k.* (Without great damage the reader may assume that K is a triangulation and, accordingly, consider the polyhedral domains of K to be simplexes.)

Proof. The theorem is obvious if $k = 0$, i.e., if T^k is a vertex of K, since every vertex of K is at the same time a vertex of K_1.

Let us suppose that Theorem 2.22 has been proved for $k \leq r$ and prove it for $k = r + 1$. Let us consider all the simplexes T'_{1h} of K_1 whose first vertex is the centroid of one of the proper faces of a given $(r + 1)$-element

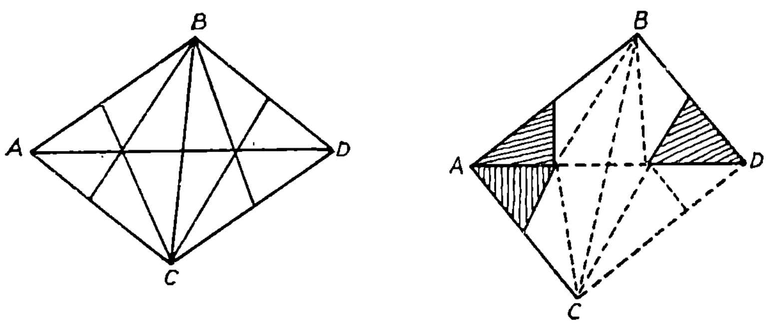
$T^{r+1} \in K$. The collection of these simplexes T'_{1h} constitutes a barycentric subdivision of the complex $\dot{T}^{r+1} = | T^{r+1} | \setminus T^{r+1}$, where, by assumption, every point of the boundary $| T^{r+1} | \setminus T^{r+1}$ of the polyhedral domain T^{r+1} is contained in some simplex T'_{1h}.

Now let $p \in T^{r+1}$; if the point p is the centroid of T^{r+1}, it is a vertex of K_1. Suppose that p is not the centroid of T^{r+1}; denote by p' the projection of the point p on $\overline{T}^{r+1} \setminus T^{r+1}$ from the centroid o of T^{r+1} and let $T'_{1h} = (e_1 \cdots e_i)$ be the simplex of K_1 containing p'. Then p is a point of the simplex $(oe_1 \cdots e_i)$ of K_1 contained in T^{r+1}, which was to be proved.

Let us combine propositions 2.21, 2.210, and 2.22 into one theorem:

2.2. *The barycentric subdivision of a polyhedral complex K is a triangulation K_1 such that every simplex of K_1 is contained in some element of K and every point of any element of K is contained in some simplex of K_1.*

EXAMPLES. 1. Fig. 85 shows a) a barycentric subdivision of a two-dimensional triangulation consisting of two triangles ABC and BCD (to-



a b

FIG. 85

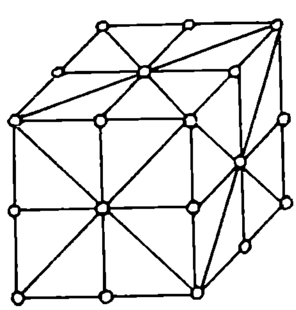


FIG. 86

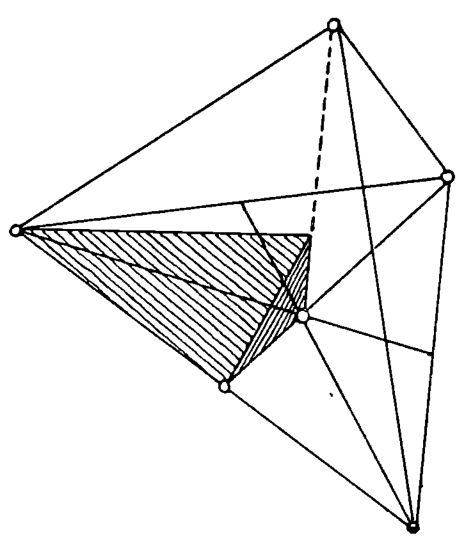


FIG. 87

gether with their sides and vertices); b) the realization of the barycentric derived of the subcomplex (nonclosed) of this triangulation consisting of the triangles ABC and BCD, the sides AB, AC, and BD and the vertices A and D.

2. Fig. 86 shows the barycentric subdivision of a polyhedral complex consisting of all the faces, edges, and vertices of a cube (the only elements shown are those turned to the observer).

3. Fig. 87 shows one of the 24 3-simplexes of the barycentric subdivision of the complex $| T^3 |$.

§2.3. The cone of a complex. Let K be a simplicial complex. In the sequel, we shall identify K with its skeleton complex.

Let o be a vertex not in K.

Let us construct the skeleton complex $<oK>$, which, by definition, consists of the following skeletons: the vertex o, all the skeletons of K, and all sets of the form $\{o, e_1, \cdots, e_r\}$, where $\{e_1, \cdots, e_r\}$ is any skeleton of K. The complex $<oK>$, as well as every complex obtained from $<oK>$ by an isomorphic mapping which leaves invariant all the elements of K (regarded as a subcomplex of $<oK>$), is called a *cone-complex* (or simply *cone*) *with vertex o and base K*.

The cone-complex of an unrestricted simplicial complex is an unrestricted simplicial complex.

The complex

$$oK = <oK> \setminus K,$$

i.e., the star of the vertex o in $<oK>$, is called an *open cone with vertex o and base K*.

The following remark, due to L. S. Pontryagin, will be essential in Chapter XIII.

2.31. *Let K be an unrestricted simplicial complex, e a vertex of K, B the outer boundary of the star $O_K e$, and T an arbitrary element of the complex B. Then the star of eT in K is the open cone with vertex e and base $O_B T$:*

$$O_K eT = eO_B T. \tag{2.31}$$

The proof consists in the immediate verification of the fact that both sides of (2.31) consist of the same simplexes. This may be left to the reader; in Fig. 88 the complex consists of the tetrahedra $(ee_3e_2e_1)$, $(ee_3e_2e_4)$ and all their faces; $T = (e_3e_2)$; the complex B consists of the triangles $(e_3e_2e_1)$ and $(e_3e_2e_4)$ and their faces, O_BT consists of the segment (e_3e_2) and

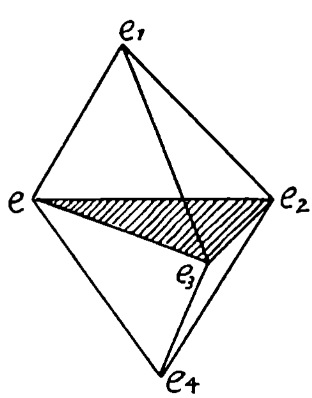


FIG. 88

the triangles $(e_3e_2e_1)$ and $(e_3e_2e_4)$; both sides of (2.31) consist of the triangle (ee_3e_2), and of the two tetrahedra $(ee_3e_2e_1)$ and $(ee_3e_2e_4)$.

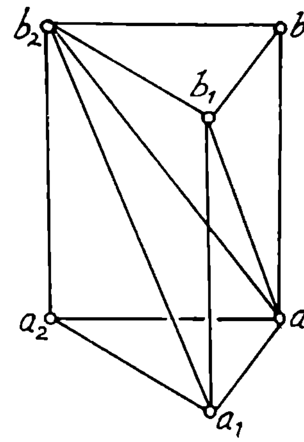


FIG. 89

§2.4. Prisms over a skeleton complex. Every subdivision of a parallelogram by a diagonal into two triangles, as well as the subdivision of a trihedral prism into tetrahedra (Fig. 89), known from textbooks of elementary geometry, are special cases of the following general construction. Let K be a nonempty finite unrestricted skeleton complex. Let us enumerate in a definite order all the vertices of K: $a_1, \cdots, a_s$. Let us consider new vertices $b_1, \cdots, b_s$ which correspond (1–1) to the vertices $a_1, \cdots, a_s$. The vertices a_i and b_i are, by definition, the vertices of a skeleton complex $K_{[01]}$ which is defined as follows. *The skeleton complex $K_{[01]}$ consists, by definition, of all nonempty subsets of sets of the form*

$$a_{i_0}, \cdots, a_{i_k}, b_{i_k}, \cdots, b_{i_r}, \tag{2.4}$$

where

$$i_0 < \cdots < i_k < \cdots < i_r$$

and

$$(a_{i_0} \cdots a_{i_k} \cdots a_{i_r}) \in K$$

[in particular, of all the skeletons $(a_{i_0} \cdots a_{i_r})$ of the complex K and of all the skeletons $(b_{i_0} \cdots b_{i_r})$ corresponding to them].

The complex $K_{[01]}$ is called the *prism over the skeleton complex K*. If K is an r-complex, $K_{[01]}$ is an $(r + 1)$-dimensional unrestricted skeleton complex.

The complex K is referred to as the *lower base of the prism $K_{[01]}$* (and is sometimes denoted by K_0). *The upper base of the prism $K_{[01]}$* is the complex $K_1 \subset K_{[01]}$ isomorphic to the complex K and consisting of all the skeletons $(b_{i_0} \cdots b_{i_r})$ corresponding to the skeletons $(a_{i_0} \cdots a_{i_r})$ of K.

§2.5. The prism spanned by a skeleton complex and its simplicial image. Let K_0 be a closed subcomplex of a finite unrestricted skeleton complex K and let S^0 be a simplicial mapping of K_0 into K which satisfies the condition: for every $T_0 \in K_0$ there is a $T \in K$ such that T_0 and S^0T_0 are faces of the simplex T. Enumerating all the vertices of K_0 in a definite order $a_1, \cdots, a_s$, let us construct the prism $K_{[01]}$ over K_0; let the corresponding vertices of the upper base K_1 of the prism $K_{[01]}$ be $b_1, \cdots, b_s$. Let us now set

$$S^{01}a_i = a_i, \qquad S^{01}b_i = S^0a_i \qquad\qquad i = 1, \cdots, s.$$

Then S^{01} maps each skeleton

$$T = (a_{i_0} \cdots a_{i_k} b_{i_k} \cdots b_{i_r}) \in K_{[01]}, \qquad i_0 < \cdots < i_r,$$

into the skeleton

$$S^{01}T = (a_{i_0} \cdots a_{i_k} S^0 a_{i_k} \cdots S^0 a_{i_r}) \in K,$$

so that the mapping S^{01} of the prism $K_{[01]}$ into K is simplicial; its image $S^{01}K_{[01]}$ is called the prism spanned by K_0 and S^0K_0 in K. (Since $a_{i_0}, \cdots, a_{i_k}, \cdots, a_{i_r}$ form a skeleton in K_0, it follows by assumption that the vertices $a_{i_0}, \cdots, a_{i_k}, \cdots, a_{i_r}, S^0a_{i_0}, \cdots, S^0a_{i_k}, \cdots, S^0a_{i_r}$ and, a fortiori, $a_{i_0}, \cdots, a_{i_k}, S^0a_{i_k}, \cdots, S^0a_{i_r}$ form a skeleton in K.)

§3. The body of a complex. Polyhedra

§3.1. Definitions.

DEFINITION 3.11. Let K be a complex which is either a triangulation or a subcomplex of a triangulation. The union of the elements of K (regarded as point sets of a given R^n) is called the *body of the complex* K and is denoted by $\| K \|$.

REMARK 1. The same definition is also applicable to a wider class of complexes, namely, to all complexes whose elements are simplexes or, in general, polyhedral domains of a given R^n.

DEFINITION 3.12. *A set which is the body of some triangulation is called a polyhedron.*

REMARK 2. It follows from 2.2 that the body of every polyhedral complex is also the body of some triangulation (for example, the body of the barycentric subdivision of a complex K) and, consequently, is a polyhedron. Hence, *polyhedra can be defined as the bodies of polyhedral complexes.*

We have the following theorem:

3.13. *Let K be a finite set of convex polyhedral domains of a given R^n bound with only one restriction: every face of an arbitrary element of K is itself an element of K. The union $\| K \|$ of all the elements of K (regarded as point sets of the given R^n) is a polyhedron.*

The proof of this theorem is left to the reader: let us note a method of proof. Let us suppose that the theorem has been proved for the case that all the elements of K are contained in the union of a finite number of m-planes (for $m = 0$ the proposition is obvious) and prove it if all the elements of K are contained in the union of a finite number of $(m + 1)$-planes $R_1^{m+1}, \cdots, R_s^{m+1}$ of R^n. Let $R_1, \cdots, R_u$ be all the planes defined as the intersection of any two of the planes R_i^{m+1} and R_j^{m+1} and also the planes carrying any m-element of K. If any of these planes, say R_i, has dimension m, let us write $R^m{}_i$ instead of R_i. But if the dimension of R_i is less than m, denote by $R^m{}_i$ any m-plane passing through R_i. Hence we

obtain m-planes $R^m{}_1, \cdots, R^m{}_u$. These m-planes divide $H = R_1{}^{m+1} \cup \cdots \cup R_s{}^{m+1}$ into a finite number of convex sets open in H; those of the domains which contain points of the set $\| K \|$ are convex polyhedral domains in $\| K \|$. Let us denote these convex polyhedral domains by $\Gamma_1, \cdots, \Gamma_r$. By the inductive hypothesis, the set $\| K \| \cap (R^m{}_1 \cup \cdots \cup R^m{}_u)$ is a polyhedron; a triangulation of this polyhedron induces a triangulation of the boundary of each of the convex polyhedral domains Γ_i. The projection of each of these triangulations from any interior point of the corresponding Γ_i leads to a triangulation of the entire set $(\Gamma_1 \cup \cdots \cup \Gamma_r) \cup (\| K \| \cap (R^m{}_1 \cup \cdots \cup R^m{}_u)) = \| K \|$, which is therefore also a polyhedron. (For a detailed proof see Alexandroff [A–H, pp. 141–143].)

From Theorem 3.13 it follows at once that

3.14. *The union of a finite number of polyhedra contained in a given R^n is a polyhedron.*

EXERCISE. Prove the following theorem:

3.15. *The intersection of two polyhedra is a polyhedron; the closure of the difference of two polyhedra is a polyhedron.*

Every triangulation whose body is a given polyhedron is called a *triangulation of the given polyhedron.*

EXAMPLE. Let T^n be a simplex. The body of the complex $| T^n |$ is obviously the set $\| \, | T^n | \, \| = \overline{T}^n$.

If K is a triangulation and $T \in K$, then $| T | \subseteq K$, and therefore $\overline{T} \subseteq \| K \|$. Hence every polyhedron is the union of the closures of a finite number of simplexes, i.e., a closed bounded set of Euclidean space. Consequently, *every polyhedron is a compactum.*

THEOREM 3.1. *If K' is a subcomplex of a polyhedral complex K, $\| K' \|$ is closed (open) in $\| K \|$ if, and only if, K' is a closed (open) subcomplex of K.*

Proof. Since open sets in $\| K \|$ are the complements of closed sets in $\| K \|$ and open subcomplexes of K are the complements of closed subcomplexes of K, it suffices to prove the assertion for closed sets and subcomplexes.

Let K' be a closed subcomplex of K. Then K' is a polyhedral complex, so that $\| K' \|$ is a polyhedron. Hence $\| K' \|$ is a compactum, and is therefore closed in every set containing it, in particular, in $\| K \|$.

Let K' be a nonclosed subcomplex of K and let $T \in K$ be a face of any element of K' not in K'; all the points of $\| T \| \subseteq \| K \|$ are limit points of $\| K' \|$ which do not belong to $\| K' \|$. Hence $\| K' \|$ is not closed.

§3.2. Star neighborhoods. Open stars.

DEFINITION 3.2. Let K be a triangulation. Let $p \in \| K \|$. The unique element $T(p)$ of the complex K which contains the point p is called the *carrier* of p in K. The set $\| O_K p \| = \| O_K T(p) \|$, which is open in $\| K \|$, is called the *star neighborhood of p in K.*

DEFINITION 3.20. The star neighborhoods of the vertices of a triangulation K (with respect to this triangulation) are called the *open stars of the triangulation K.*

Hence the open stars of a triangulation are open subsets of its body.

THEOREM 3.21. *The open stars* $\| O_K e \|$ *of a triangulation K cover the polyhedron* $\| K \|$.

Indeed, let $p \in \| K \|$, let T be the carrier of p, and let e be a vertex of T. Obviously,

$$p \in T \in O_K e,$$

$$p \in T \subseteq \| O_K e \|.$$

This proves the theorem.

THEOREM 3.22. *The intersection of the bodies of a finite number of subcomplexes of a polyhedral complex K is the body of the intersection of these subcomplexes.*

It suffices to prove this theorem for two subcomplexes $K_1 \subseteq K$ and $K_2 \subseteq K$, i.e., to prove that

$$\| K_1 \cap K_2 \| = \| K_1 \| \cap \| K_2 \|.$$

If $p \in T \subseteq K_1 \cap K_2$, $p \in \| K_1 \| \cap \| K_2 \|$, i.e., $\| K_1 \cap K_2 \| \subseteq \| K_1 \| \cap \| K_2 \|$.

Conversely, let $p \in \| K_1 \| \cap \| K_2 \|$. Since the unique element of K containing p is the carrier $T(p)$ of p, $T(p) \in K_1$, $T(p) \in K_2$, i.e., $T(p) \in K_1 \cap K_2$, so that $p \in \| T(p) \| \subseteq \| K_1 \cap K_2 \|$. Hence $\| K_1 \| \cap \| K_2 \| \subseteq \| K_1 \cap K_2 \|$.

3.23. The open stars $\| O_K e_0 \|, \cdots, \| O_K e_r \|$ of a triangulation K intersect if, and only if, K contains the simplex $(e_0 \cdots e_r)$.

This assertion follows from 3.22 and from the Corollary to Theorem 1.89.

In other words:

3.24. *Every triangulation is the nerve of its system of open stars.*

§3.3. Simplicial mappings of triangulated polyhedra. A simplicial mapping S_α^β of a triangulation K_β into a triangulation K_α induces a continuous mapping $\tilde{S}_\alpha^\beta$ of the polyhedron $\| K_\beta \|$ into the polyhedron $\| K_\alpha \|$ in the following way. Let $T_\beta = (e_0 \cdots e_r)$ be any simplex of the complex K_β. The mapping $\tilde{S}_\alpha^\beta = S_\alpha^\beta$ is defined on the vertices of the simplex T_β and this yields an affine mapping $\tilde{S}_\alpha^\beta$ of the simplex T_β onto the simplex $S_\alpha^\beta T_\beta \in K_\alpha$ with vertices $S_\alpha^\beta e_0$, $S_\alpha^\beta e_1$, $\cdots$, $S_\alpha^\beta e_r$. The required mapping $\tilde{S}_\alpha^\beta$ of the polyhedron $\| K_\beta \|$ into $\| K_\alpha \|$ is therefore defined in every simplex $T_\beta \in K_\beta$. The continuity of the mapping $\tilde{S}_\alpha^\beta$ is easily proved: let $p \in \| K_\beta \|$, let T_β be the carrier of the point p, and let $p = \lim p_n$. Without loss of generality, we may suppose that all the p_n are contained in the

same $T'_\beta \geq T_\beta$. The barycentric coordinates of p_n with respect to the skeleton of the simplex T'_β approach the barycentric coordinates of p. Therefore the weights assigned to the images of the vertices of T'_β to obtain the point $\tilde{S}_\alpha{}^\beta p_n$ approach the weights defining the point $\tilde{S}_\alpha{}^\beta p$, whence $\lim \tilde{S}_\alpha{}^\beta p_n = \tilde{S}_\alpha{}^\beta p$. The mapping $\tilde{S}_\alpha{}^\beta$ is called a *simplicial mapping of the polyhedron* $\| K_\beta \|$ *into the polyhedron* $\| K_\alpha \|$ induced by the simplicial mapping $S_\alpha{}^\beta$ of the complex K_β into K_α.

If a simplicial mapping $S_\alpha{}^\beta$ of a complex K_β onto a complex K_α is $(1-1)$, the mapping of the polyhedron $\| K_\beta \|$ onto the polyhedron $\| K_\alpha \|$ is also $(1-1)$ and therefore topological. Whence it follows that:

THEOREM 3.31. *If K_β and K_α are isomorphic triangulations, the polyhedra $\| K_\beta \|$ and $\|K_\alpha \|$ are homeomorphic.*

Since every n-dimensional triangulation is isomorphic to a triangulation in R^{2n+1}, we have the following proposition:

THEOREM 3.32. *Every n-dimensional polyhedron is homeomorphic to a polyhedron in R^{2n+1}.*

§4. Subdivisions of polyhedral complexes

§4.1. Definition of a subdivision.

DEFINITION 4.11. Let K be an arbitrary polyhedral complex. (The reader may assume that all the polyhedral complexes mentioned in this section are triangulations.) *A subdivision of the complex K* is any polyhedral complex K_α satisfying the following conditions:

1. The body of the complex K_α coincides with the body of the complex K.
2. Every element of the complex K_α, considered as a point set, is contained in some element of the complex K.

The elements of K are mutually disjoint. Therefore, if K_α is a subdivision of K, every element of K_α is contained in just one element of K. *The unique element T_j of K containing a given element $T_{\alpha i}$ of K_α is called the carrier of $T_{\alpha i}$ in K.*

If $T_{\alpha i} \in K_\alpha$ has as its carrier $T_j \in K$ and T_h is an element of K distinct from T_j, then $T_{\alpha i} \cap T_h = 0$. Indeed, if $T_{\alpha i}$ and T_h had a common point, then T_j and T_h would have a common point. This is impossible, since T_j and T_h are distinct elements of K.

Hence

THEOREM 4.12. *If K_α is a subdivision of a complex K, every element $T_{\alpha i}$ of the complex K_α is contained in exactly one element of the complex K—in the carrier of the element $T_{\alpha i}$—and does not intersect any other element of K.*

4.11 and 2.2 imply that the barycentric subdivision of a polyhedral complex K defined in 2.2 is, in fact, a subdivision in the sense just introduced.

Since a barycentric subdivision of an arbitrary polyhedral complex consists of simplexes, it follows that:

4.13. *Every polyhedral complex numbers triangulations among its subdivisions.*

DEFINITION 4.14. A subdivision of a simplex T^n is the complex consisting of all the elements of a subdivision of the complex $| T^n |$ which are contained in T^n.

§4.2. Successive barycentric subdivisions. Let K_1 be the barycentric subdivision of a complex K, K_2 the barycentric subdivision of the complex K_1; in general, let K_ν be the barycentric subdivision of the complex $K_{\nu-1}$. The complex K_ν is called the barycentric subdivision of order ν of K.

If the diameter of an n-simplex T^n is d, the simplexes of the barycentric subdivision of the complex $| T^n |$ have diameter $\leq (n/n + 1)\, d$ (see Appendix 1, Theorem 4.2).

Hence it follows that if all the simplexes of a triangulation K are of diameter $\leq d$, then all the simplexes of the complex K_ν have diameter $\leq (n/n + 1)^\nu d$.

Since $\lim_{\nu\to\infty}(n/n + 1)^\nu = 0$, we have the following result:

THEOREM 4.21. *Every polyhedral complex has subdivisions of arbitrarily small mesh, i.e., subdivisions consisting of simplexes whose diameters are less than an arbitrary preassigned positive number.*

COROLLARY. Every polyhedron has triangulations of arbitrarily small mesh.

From 4.21 it follows that

4.22. *If a polyhedron Φ is the body of an n-dimensional triangulation K,* $\dim.\Phi \leq n$.

REMARK. See I, 8.4. In the following section (Theorem 5.34) Theorem 4.22 will be proved again; the second proof is simpler because it is based directly on the definition of dimension (I, 8.42), and not on the comparatively complicated Theorem 8.44 of the same chapter.

Proof of Theorem 4.22. Let K_1 be a subdivision of the triangulation K, all of whose simplexes are of diameter $<\epsilon$. Then the open stars of the triangulation K_1 form an open 2ϵ-covering of the polyhedron Φ, whose nerve is K_1. Theorem 4.22 now follows from I, 8.44.

§4.3. Central and elementary subdivisions of complexes. We have considered in detail the barycentric subdivision of complexes, since this is the most important type of subdivision. However, barycentric subdivisions are not the very simplest.

a) Let K^n be a polyhedral complex. Denote by K^{n-1} the complex composed of all the elements of K^n of dimension $\leq n - 1$, and assume as given a subdivision $K_\alpha{}^{n-1}$ of K^{n-1}. The subdivision $K_\alpha{}^{n-1}$ of K^{n-1} induces a subdivision $K^n{}_\alpha$ of K^n called the *central subdivision of K^n relative to the given subdivision $K_\alpha{}^{n-1}$ of K^{n-1}*: let o_i be the centroid of $T^n{}_i \in K^n$ and consider

the polyhedral domains which are the projections from o_i of the elements of the complex K_α^{n-1} contained in $\bar{T}^n_i \setminus T^n_i$. These polyhedral domains, the points o_i, and the elements of the complex K_α^{n-1} are, by definition, the elements of the complex K^n_α.

The proof of the fact that K^n_α is, in fact, a subdivision of K^n offers no difficulty.

The barycentric subdivision of an arbitrary polyhedral complex K^n is reduced to a series of central subdivisions in the following way. The barycentric subdivision K^0_1 of the 0-complex K^0, consisting of the vertices of K^n, is the complex K^0 itself. Let us suppose that the barycentric subdivision K^r_1 of the complex K^r, consisting of all the simplexes of K^n of dimension $\leq r$, has already been constructed. Then we obtain the barycentric subdivision K_1^{r+1} of the complex K^{r+1} as the central subdivision of K^{r+1} with respect to the barycentric subdivision K^r_1 of K^r.

b) *Elementary subdivision.* The central subdivision of the complex $| T^n |$ relative to $| T^n | \setminus T^n$ is called simply the central subdivision of $| T^n |$; the simplexes of this subdivision contained in T^n form a central subdivision of the simplex T^n. The central subdivision of T^n or $| T^n |$, respectively, is also called the elementary subdivision relative to T^n. The elementary subdivision of T^n relative to $T^p < T^n$ consists, by definition, of all the simplexes $(T^r_i T^{n-p-1})$, where T^r_i is any element of the central subdivision of T^p and T^{n-p-1} is the face of the simplex T^n opposite the simplex T^p; here, $(T^r_i T^{n-p-1})$, as always, denotes the simplex whose skeleton is the union of the skeletons of the simplexes T^r_i and T^{n-p-1}, i.e., the combinatorial sum of T^r_i and T^{n-p-1}.

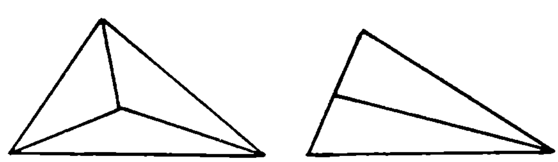


FIG. 90

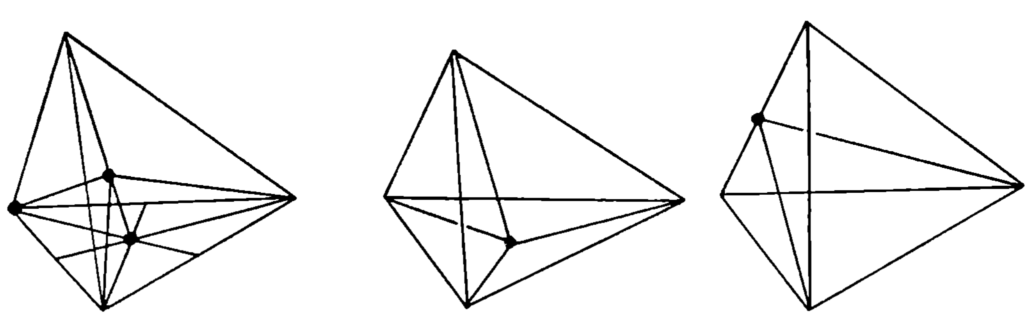


FIG. 91

The elementary subdivision of a triangulation K relative to a simplex $T^p \in K$ is, by definition, the subdivision obtained by replacing each simplex $T \in O_K T^p$ by its elementary subdivision relative to T^p and leaving all the remaining simplexes (i.e., all the simplexes of the complex $K \setminus O_K T^p$) unchanged.

Figs. 90 and 91 show various cases of the elementary subdivisions of 2- and 3-simplexes.

The definition of an elementary subdivision immediately implies the following remarks which are required in Chapters VII and X.

REMARK 1. Let $T^{n-p-1} = (e_{p+1} \cdots e_n)$ be the face of the simplex

$$T^n = (e_0 \cdots e_p e_{p+1} \cdots e_n)$$

opposite the face $T^p = (e_0 \cdots e_p)$. Let e be the center of the simplex T^p. Then the simplexes of the elementary subdivision of T^n relative to T^p are all the simplexes of the form

$$(ee_{i_0} \cdots e_{i_r} e_{p+1} \cdots e_n), \qquad i_0 < \cdots < i_r \leq p;\ r \leq p - 1,$$

and only these.

REMARK 2. Let V^n be the elementary subdivision of a simplex T^n relative to one of its faces T^p. In order that a subset E of the set of all vertices of the complex $| V^n |$ be a skeleton of $| V^n |$ it is necessary and sufficient that E not contain the skeleton of the simplex T^p; among the skeletons of $| V^n |$ the skeletons of V^n are characterized by the fact that they contain the vertex e and all the vertices of the face $T^{n-p-1} < T^n$ opposite the face T^p.

REMARK 3. Let us preserve the notation of the preceding remark. If $r \leq n - 1$, an r-simplex T^r of the complex V^n cannot have among its vertices more than $p - 1$ vertices of the simplex T^p.

Indeed, in the contrary case, the simplex T^r, having among its vertices the $n - p$ vertices $e_{p+1}, \cdots, e_n$ of the simplex T^{n-p-1} and also the vertex e, would have in all at least $p + n - p + 1 = n + 1$ vertices; this is impossible, since $r \leq n - 1$.

Let us show that the barycentric subdivision of a triangulation K is reducible to a series of elementary subdivisions. To this end, let us first perform an elementary subdivision on all the original simplexes of the complex K relative to themselves. Then simplexes of the form $T^n = (e_0 \cdots e_n)$ give rise to simplexes (for the notation see the end of Remark 1 of 0.2) $T^n{}_{i_1} = (oe_0 \cdots \hat{e}_{i_1} \cdots e_n)$. Next, let us perform an elementary subdivision on each of these simplexes relative to the faces $(e_0 \cdots \hat{e}_{i_1} \cdots e_n)$. The resulting simplexes have the form

$$T^n{}_{i_1 i_2} = (oo_{i_1} e_0 \cdots \hat{e}_{i_1} \cdots \hat{e}_{i_2} \cdots e_n).$$

After the elementary subdivision of these simplexes relative to the faces $(e_0 \cdots \hat{e}_{i_1} \cdots \hat{e}_{i_2} \cdots e_n)$, the simplexes $T^n{}_{i_1 i_2 i_3}$ are obtained, whose first three vertices are o, o_{i_1}, $o_{i_1 i_2}$, and whose remaining vertices are vertices of T^n. [If T^r is a face of the simplex $T^n = (e_0 \cdots e_n)$ and $e_{i_1}, \cdots, e_{i_k}$ are the vertices of T^n which are not the vertices of T^r, then $o_{i_1 \cdots i_k}$ is the centroid of the simplex T^r.]

Continuing the indicated process [the simplex $T^n{}_{i_1 \cdots i_k}$ is defined as $T^n{}_{i_1 \cdots i_k} = (oo_{i_1} \cdots o_{i_1 \cdots i_k} e_{i_{k+1}} \cdots e_{i_n})$, we finally obtain simplexes of the form $(oo_{i_1} \cdots o_{i_1 \cdots i_n})$ which, together with their faces, form a barycentric subdivision of the simplex T^n.

§4.4. Subdivisions of nonclosed subcomplexes of polyhedral complexes. Let K be a subcomplex of a polyhedral complex K'. Then $|K|$ is a polyhedral complex. Every complex consisting of all those simplexes of any subdivision K_α of $|K|$ which are contained in elements of K (i.e., have these elements as their carriers) is called a subdivision of K.

In particular, if K_α is the barycentric (elementary) subdivision of $|K|$, then the subcomplex of K_α consisting of all the simplexes of K_α having simplexes of K as their carriers is called the barycentric (elementary) subdivision of K.

REMARK 1. It follows from this definition that one can speak of the elementary subdivision of a complex K relative to a simplex not belonging to this complex (i.e., relative to a simplex $T \in |K| \setminus K$).

REMARK 2. If K is a nonclosed subcomplex of a simplicial complex, the barycentric subdivision of K will not be an unrestricted complex and therefore cannot be isomorphic to the barycentric derived of K.

§5. Barycentric stars

§5.1. Barycentric stars. Let K be an n-dimensional triangulation, and let K_1 be its barycentric subdivision. Let

$$T_1, T_2, \cdots, T_s$$

be all the simplexes of K, and let

$$T_1 = e_1, \cdots, T_u = e_u, \qquad u = \rho_0,$$

be the 0-simplexes of K, i.e., the vertices of K. The vertices of K_1 are the centers

$$e_{11}, e_{12}, \cdots, e_{1s}$$

of the simplexes

$$T_1, T_2, \cdots, T_s \ (e_{1i} = e_i \quad \text{for} \quad i \leq \rho_0).$$

We shall write $e_{1i} > e_{ij}$ if $T_i > T_j$, so that the simplexes of K_1 have the form

$$(e_{1i_0} > e_{1i_1} > \cdots > e_{1i_r}).$$

DEFINITION 5.1. Let $T_i \in K$. Let us call the subcomplex of K_1 consisting of all the simplexes of K_1 whose last vertex is the center e_{1i} of T_i the *barycentric star dual to* T_i (or simply the *dual* of T_i). We shall denote the star dual to T_i by T^*_i, and will refer to the simplex T_i itself as the *dual* of (the barycentric star) T^*_i. The *outer boundary of the barycentric star* T^*_i is, by definition, the complex $\dot{T}^*_i = |\, T^*_i \,| \setminus T^*_i$, i.e., the set of all faces of the simplexes of T^*_i not in T^*_i. The barycentric stars dual to the vertices e_i of K are known as the *major stars*. These are the barycentric stars $T^*_1, \cdots, T^*_r$, $r = \rho_0$ (Fig. 92).

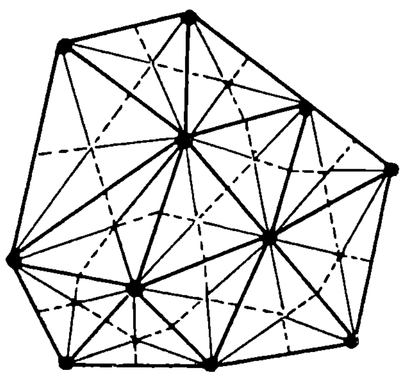


FIG. 92

5.11. If $T_i < T_j$, the dimension of the complex T^*_i is greater than the dimension of the complex T^*_j.

In fact, let the dimension of T^*_j be r, and let

$$(e_{1j_1} > e_{1j_2} > \cdots > e_{1j_r} > e_{1j})$$

be an r-simplex of T^*_j. Then the simplex

$$(e_{1j_1} > e_{1j_2} > \cdots > e_{1j_r} > e_{1j} > e_{1i})$$

is an $(r + 1)$-simplex of T^*_i (Fig. 92).

§5.2. The dual complex of a triangulation.

DEFINITION 5.2. The set K^* whose elements are the complexes T^*_i is called the *dual* (or *barycentric*) *complex* of the triangulation K; the set K^* is in this connection partially ordered by putting

$$T^*_i < T^*_j \quad \text{if} \quad T_i > T_j .$$

This order and the dimensions of the complexes T^*_i (as subcomplexes of K) turn K^* into an abstract complex (see 1.7). Hence the term "dual complex" is fully justified and we may refer to open and closed subcomplexes of K^*.

We shall derive a series of simple but important properties of K^*.

5.21. $T^*_1 \cup \cdots \cup T^*_s = K_1$.

Indeed, if $T_{1h} \in K_1$ and e_{1i} is the last vertex of T_{1h}, then $T_{1h} \in T^*_i$.

5.21 implies

5.210. $\| T^*_1 \| \cup \cdots \cup \| T^*_s \| = \| K_1 \| = \| K \|$.

5.22. If $i \neq j$, then $T^*_i \cap T^*_j = 0$.

Indeed, if $T_{1h} \in T^*_i \cap T^*_j$, the simplex $T_{1h} \in K_1$ would have two last vertices e_{1i} and e_{1j}, which is impossible.

5.220. If $i \neq j$, then $\| T^*_i \| \cap \| T^*_j \| = 0$.

This follows from 5.22 and Theorem 3.22.

5.23. Every non-major barycentric star is less than at least one major star.

Indeed, if $e_k < e_{1i}$, then $T^*_k > T^*_i$.

5.24. The outer boundary of a barycentric star T^*_i is the union of the barycentric stars less than T^*_i.

Proof. It follows from the definition of the complex $\dot{T}^*_i \subset K_1$ that:

5.240. The complex $\dot{T}^*_i$ consists of all the simplexes of the form $T_{1h} = (e_{1j_0} > e_{1j_1} > \cdots e_{1j_r})$, where $e_{1j_r} > e_{1i}$.

Therefore, if

$$T_{1h} = (e_{1j_0} > \cdots > e_{1j_r}) \in \dot{T}^*_i,$$

then

$$e_{1j_r} > e_{1i} \quad \text{and} \quad T_{1h} \in T^*_{j_r} < T^*_i.$$

Conversely, if $T_{1h} \in T^*_j < T^*_i$, then

$$T_{1h} = (e_{1j_0} > \cdots e_{1j_r} > e_{1j}), e_{1j} > e_{1i},$$

and $T_{1h} \in \dot{T}^*_i$.

5.25. The *combinatorial closure* $| T^*_i |$ of the barycentric star T^*_i in the complex K_1, i.e., the subcomplex of K_1 consisting of all the faces of the simplexes of T^*_i, can be written in the form

$$| T^*_i | = T^*_i \cup \dot{T}^*_i. \tag{5.25}$$

5.240 implies

5.250. The complex $| T^*_i |$ consists of all the simplexes $T_{1h} \in K_1$ of the form $T_{1h} = (e_{1j_0} > e_{1j_1} > \cdots > e_{1j_r})$, where $e_{1j_r} \geq e_{1i}$.

Whence it follows that:

5.26. If $T^*_i < T^*_j$, then $| T^*_i | \subset | T^*_j |$.

From 5.21, 5.23, 5.24, and 5.25 we infer

5.27. The union of the combinatorial closures of the major barycentric stars of the complex K is the complex K_1.

THEOREM 5.28. *A necessary and sufficient condition that*

$$| T^*_{i_0} | \cap \cdots \cap | T^*_{i_r} | \neq 0$$

is that the simplexes $T_{i_0}, \cdots, T_{i_r}$ be faces of a simplex of K; in that case

$$| T^*_{i_0} | \cap \cdots \cap | T^*_{i_r} | = | T^*_i |,$$

where T_i is the combinatorial sum of the simplexes $T_{i_0}, \cdots, T_{i_r}$.

Proof. a) Let

$$(e_{1j_0} > \cdots > e_{1j_p}) \in | T^*_{i_0} | \cap \cdots \cap | T^*_{i_r} |;$$

then, by 5.250

$$e_{1j_p} \geq e_{1i_k} \qquad (0 \leq k \leq r),$$

i.e., all the simplexes $T_{i_0}, \cdots, T_{i_r}$ are faces of the simplex T_{j_p}; hence the combinatorial sum T_i of the simplexes $T_{i_0}, \cdots, T_{i_r}$ exists and is a face of the simplex T_{j_p}:

$$T_{j_p} \geq T_i .$$

Therefore,

$$T^*_{j_p} \leq T^*_i , (e_{1j_0} > \cdots > e_{1j_p}) \in T^*_{j_p} \subseteq | T^*_i |.$$

Hence

$$| T^*_{i_0} | \cap \cdots \cap | T^*_{i_r} | \subseteq | T^*_i |.$$

b) To prove the converse inclusion we note that, according to the definition of combinatorial sum,

$$T_i \geq T_{i_k} , \qquad k = 0, 1, \cdots, r,$$

so that

$$T^*_i \leq T^*_{i_k} , | T^*_i | \subseteq | T^*_{i_k} |,$$

$$| T^*_i | \subseteq | T^*_{i_0} | \cap \cdots \cap | T^*_{i_r} |.$$

This proves 5.28.

In particular, the *combinatorial closures of the major stars*

$$T^*_{i_0} , \cdots , T^*_{i_r} , \qquad i_0 , \cdots , i_r \leq \rho_0 ,$$

intersect if, and only if, the simplex

$$(e_{i_0} \cdots e_{i_r})$$

is in K.

In other words:

THEOREM 5.29. *The nerve of the system of combinatorial closures of the major barycentric stars of a complex* K *is the complex* K *itself* (Fig. 92).

§5.3. Closed barycentric stars.

DEFINITION 5.31. The body of the combinatorial closure $| T^*_i |$, $i \leq \rho_0$, of a major barycentric star T^*_i dual to a vertex e_i of a complex K is called the *closed barycentric star* dual to the vertex e_i; the vertex e_i is called the *center* of the closed barycentric star.

Hence, *closed barycentric stars are polyhedra.*

5.27 implies that:

5.32. $\| \, | T^*_1 | \, \| \cup \cdots \cup \| \, | T^*_u | \, \| = \| K_1 \| = \| K \|$, where $u = \rho_0$.

It follows from 5.29 and 3.21 that:

5.33. *The closed barycentric stars of a complex K form a closed covering of the polyhedron $\| K \|$, called the barycentric covering of $\| K \|$ dual to its triangulation K. The nerve of this covering is the complex K.*

Since the simplexes and hence the closed barycentric stars of a complex K can be assumed to be of arbitrarily small diameter, it follows from 5.33 that:

THEOREM 5.34. *The body of every n-dimensional triangulation has a closed ϵ-covering of order $n + 1$ for arbitrary ϵ.*

§5.4. The subcomplexes of the complex K*; their bodies and barycentric subdivisions. Let K be any triangulation, K_1 its barycentric subdivision, and K^* its dual complex. Let K_0 be any subcomplex of K. Denote by K^*_0 the subcomplex of K^* consisting of all the barycentric stars of K dual to the elements of K_0.

Conversely, to every subcomplex K^*_0 of K^* there corresponds the subcomplex $K_0 \subseteq K$ consisting of all the elements of K dual to the elements of K^*_0. *The complexes K_0 and K^*_0 are said to be dual subcomplexes of K and K^*, respectively.*

It follows that K_0 and K^*_0 are also dual to each other as partially ordered sets (see I, 6.4). Hence if K_0 is a closed (open) subcomplex of K, K^*_0 is an open (closed) subcomplex of K^*. The elements of K^*_0 are barycentric stars, i.e., certain subcomplexes of K_1; the union of all the barycentric stars which appear as elements of K^*_0 is a subcomplex K^*_{01} of K_1, called the *barycentric subdivision* of K^*_0.

The body of K^*_{01} is, by definition, the body $\| K^*_0 \|$ of K^*_0 (obviously, $\| K^*_0 \|$ is the union of the bodies of the barycentric stars which are the elements of K^*_0).

5.41. If K^*_0 is a closed subcomplex of K^*, K^*_{01} is a closed subcomplex of K_1.

Indeed, if $T^* \in K^*_0$, then, since K^*_0 is closed in K^*, the combinatorial closure of $T^* \in K^*_0$ in K^* is contained in K^*_0 and consequently, by 5.25 and 5.24,

$$| T^* | \subseteq K^*_{01} .$$

This proves 5.41.

COROLLARY. The body of every closed subcomplex K^*_0 of K^* is a polyhedron.

REMARK. If K_0 is a closed subcomplex of a complex K and K^*_0 is the subcomplex of K^* dual to K_0, then $K^* \setminus K^*_0$ is a closed subcomplex of K^* and hence $\| K^* \| \setminus \| K^*_0 \|$ is a polyhedron.

In conclusion, we shall prove the following proposition which is required in Chapter X:

5.42. *The closure of a simplex T of a triangulation K is contained in the union of the closed barycentric stars dual to the vertices of T and does not intersect any closed barycentric star whose center is not a vertex of T.*

Proof. In the course of this proof p will denote a point of the closed simplex $\overline{T}$ and

$$T_{1i} = (e_{1i_0} > \cdots > e_{1i_r})$$

the carrier of the point p in the complex K_1. It follows that T_{i_0} *is the carrier of* p in K; therefore T_{i_0} *is a face of* T. A fortiori,

$$T_{i_r} \leq T,$$

so that *all the vertices of T_{i_r} are also vertices of T.*

Let $e_h = e_{1h}$ be any vertex of T_{i_r}; the simplex

$$T_{1i} = (e_{1i_0} > \cdots > e_{1i_r})$$

is a face of the simplex

$$(e_{1i_0} > \cdots > e_{1i_r} > \cdots > e_{1h})$$

which is in the barycentric star dual to e_h and p is contained in the closed barycentric star with center e_h. But e_h, as a vertex of T_{i_r}, is a vertex of T. This proves the first half of Theorem 5.42.

To prove the second half of the theorem we note that if p is contained in the closed barycentric star dual to a vertex $e_j \in K$, then T_{1i} is a face of some simplex of K_1 having e_j as a vertex; since

$$T_{1i} = (e_{i_0} > \cdots > e_{1i_r})$$

and e_j is a vertex of K, e_j is a vertex of T_{i_r}. Hence e_j is a vertex of T, q.e.d.

REMARK. Since closed simplexes and closed barycentric stars are compacta, a closed simplex $\overline{T}$ has a nonempty intersection only with the closed barycentric stars dual to its vertices and has a positive distance from all other closed barycentric stars. Since, on the other hand, the closed barycentric stars cover the polyhedron $\| K \|$, some neighborhood of the closed simplex $\overline{T}$ is contained in the union of the closed barycentric stars dual to the vertices of T and does not intersect any of the other closed barycentric stars. Since K is a finite complex, we have

5.43. If K is a triangulation of a polyhedron $\| K \|$ there is an

$$\epsilon = \epsilon(K) > 0$$

such that an ϵ-neighborhood of an arbitrary closed simplex $\overline{T}$ of K is contained in the union of the closed barycentric stars dual to the vertices of T and does not intersect any of the other closed barycentric stars.

§6. Topological complexes and topological polyhedra

§6.1. Definitions. The definition of a polyhedron (Def. 3.12) is not topologically invariant; the topological image of a polyhedron is in general not a polyhedron.

It is natural to consider the topological images of complexes and polyhedra. We introduce the following definitions:

DEFINITION 6.11. A compactum homeomorphic to a polyhedron is called a *topological polyhedron*.

A topological mapping of a polyhedron P onto a topological polyhedron $\mathcal{P}$ maps the simplexes T_i of a given triangulation K of P onto subsets of $\mathcal{P}$. The collection of such subsets forms a so called *topological complex*. Thus we arrive at the following definition:

6.12. A finite system $\mathcal{K}$ of subsets $\mathfrak{T}_i$, $i = 1, 2, \cdots, s$, of a topological space R is called a *topological complex* if the union

$$\| \mathcal{K} \| = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_s$$

can be mapped topologically onto a polyhedron $P = \| K \|$ in such a way that under this mapping the sets $\mathfrak{T}_i$ correspond (1–1) to the simplexes T_i of some triangulation K of P (i.e., the image of each of the sets $\mathfrak{T}_i$ is a simplex T_i and, conversely, the inverse image of each of the simplexes T_i is a set $\mathfrak{T}_i$).

The union of the elements ("topological simplexes") $\mathfrak{T}_i$ of a topological complex $\mathcal{K}$ is a topological polyhedron $\| \mathcal{K} \|$, the *body* of $\mathcal{K}$; $\mathcal{K}$ is called a *topological triangulation* of $\| \mathcal{K} \|$.

Examples of topological complexes are shown in Figs. 93 and 94.

REMARK 1. Various, but necessarily isomorphic, triangulations K correspond to a topological complex $\mathcal{K}$. The dimension number of $\mathcal{K}$ is the dimension number of a triangulation K isomorphic to $\mathcal{K}$. We shall prove in Chapter V that the dimension (I, 8.4) of a polyhedron is equal to the

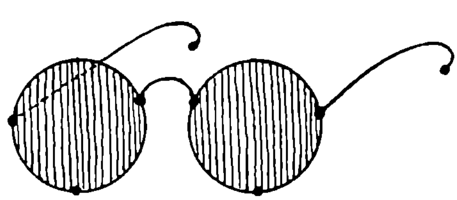


FIG. 93

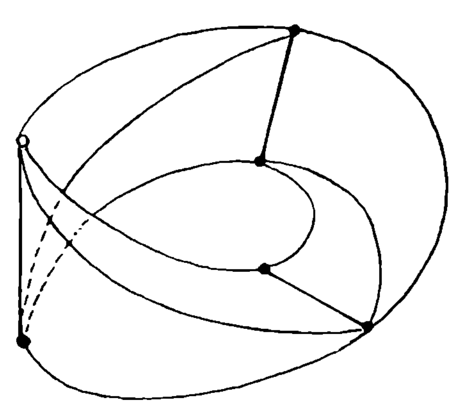


FIG. 94

dimension number of any of its triangulations. Hence the dimension number of K (equal to the dimension number of $\mathcal{K}$) is the same as the dimension of the two homeomorphic compacta $\| K \|$ and $\| \mathcal{K} \|$. Consequently, assuming the fundamental theorem of Chapter V, we may state the following proposition:

THEOREM 6.1. *All the topological triangulations of a given topological polyhedron have the same dimension number, equal to the dimension of the polyhedron.*

REMARK 2. Every topological triangulation of a topological polyhedron $\mathcal{P}$ is isomorphic to some (in general, topological) triangulation of a *given fixed* polyhedron P homeomorphic to $\mathcal{P}$. But, on the other hand, every topological triangulation of $\mathcal{P}$ is isomorphic to a triangulation of some (arbitrarily chosen) polyhedron P homeomorphic to $\mathcal{P}$. Hence the set of all combinatorial types of topological triangulations of a topological polyhedron may be defined in two ways:

a) as the collection of *topological triangulations of a single arbitrarily chosen polyhedron* P homeomorphic to $\mathcal{P}$;

b) as the collection of triangulations of *different polyhedra* homeomorphic to $\mathcal{P}$.

REMARK 3. Purely combinatorial conditions (i.e., conditions independent of the notion of continuity) which two triangulations must satisfy in order to have homeomorphic bodies are not known as yet. In particular, the so called *fundamental hypothesis* ("Hauptvermutung") *of combinatorial topology* has not yet been proved (see Moise [a], whose papers appeared several years after this book (Trans.)). This hypothesis asserts that *any two triangulations of homeomorphic polyhedra have isomorphic subdivisions.*

§6.2. n-manifolds. A *closed polyhedral manifold* is a polyhedron which is at the same time a topological n-manifold (I, 5.3); topological polyhedra which are topological manifolds are usually called simply *closed manifolds.* It is not yet known whether the class of closed manifolds defined in this way coincides with the class of all topological closed manifolds: it is not known whether there exists a closed topological manifold which is not a topological polyhedron.

A closed 1-manifold is homeomorphic to a circumference. The closed 2-manifolds are none other than the closed surfaces discussed in Chapter III.

The simplest example of a closed n-manifold is the n-sphere S^n. It can be triangulated by taking an $(n + 1)$-simplex inside the sphere $S^n \subset R^{n+1}$ and projecting its boundary onto the sphere from some point in its interior.

The three-dimensional torus (i.e., the topological product of three circumferences, I, 2.6) is also a manifold: to triangulate the three-dimensional torus it suffices to take a second order barycentric subdivision of a

three-dimensional cube and to identify opposite faces of the cube (I, 5.2, Example 6).

In the same way it is easy to construct a triangulation of the topological product of a sphere and a circumference: it is merely necessary to use the model of this product given in I, 5.2, Example 7.

To construct a triangulation of the projective n-space let us first define a regular $(n + 1)$-dimensional octahedron in R^{n+1} as a convex polyhedral domain with $2(n + 1)$ vertices e_k, e'_k, $k = 1, 2, \cdots, n + 1$, where e_k, e'_k, respectively, have all coordinates equal to zero except the k^{th} which is equal to 1, -1, respectively. The boundary of an $(n + 1)$-dimensional octahedron, as the boundary of an $(n + 1)$-dimensional convex polyhedral domain, is homeomorphic to the n-sphere S^n [in the given special case this is easy to prove directly, e.g., by complete induction on the number of dimensions: a regular $(n + 1)$-dimensional octahedron is the union of two pyramids with vertices $(0, 0, \cdots, 1)$ and $(0, 0, \cdots, -1)$ constructed on an n-dimensional regular octahedron] and is given in a triangulation K^n possessing the property of central symmetry relative to the origin of coordinates. This means that every element of K^n is mapped by the symmetric transformation

$$x'_k = -x_k, \qquad k = 1, 2, \cdots, n + 1,$$

of R^{n+1} into some other element of K^n.

A second order barycentric subdivision [one could get along more simply, i.e., with a first order barycentric subdivision; a second order subdivision is convenient for some special purposes (see the triangulation $K^n{}_1$ mentioned below)] $K^n{}_{(2)}$ of K^n obviously also possesses the property of central symmetry. Identifying symmetric elements of $K^n{}_{(2)}$, we obtain the required triangulation $K^n{}_0$ of the projective space P^n.

In addition to $K^n{}_0$ we shall need (in VIII, 4.4) still another triangulation of the projective space P^n, which will be denoted by $K^n{}_1$. It is obtained by taking a second order barycentric subdivision of an n-dimensional octahedron and its boundary and identifying elements of the triangulation of the boundary (i.e., of $K^{n-1}{}_{(2)}$) symmetric relative to the origin of coordinates.

The simplest triangulation of the projective plane P^2 is obtained by identifying symmetric elements (faces, edges, and vertices) of a regular icosahedron (a polyhedral domain with twenty faces).

§7. Connectedness of complexes

[All the results of this section (except those of 7.3) are true for abstract complexes, even for all partially ordered sets (discrete spaces), and are special cases of the corresponding theorems for T_0-spaces (I, 3).]

§7.1. Connected complexes. Components. A complex K is said to be *connected* if it is not the union of two nonempty disjoint closed subcomplexes.

REMARK. If $K = K' \cup K''$, $K' \cap K'' = 0$ and K', K'' are closed, then $K' = K \setminus K''$, $K'' = K \setminus K'$, and consequently K' and K'' are open. Therefore, the definition of connectedness can also be stated in either of the following ways:

K is a connected complex if it is not the union of two nonempty disjoint open subcomplexes.

K is a connected complex if no proper subcomplex of K is both closed and open in K.

7.11. If a connected subcomplex K_0 of a complex K is contained in the union of two closed (open) disjoint subcomplexes K_1 and K_2 of K, it is contained either in K_1 or in K_2.

Inded, if

$$K_0 \cap K_1 \neq 0 \neq K_0 \cap K_2,$$

then $K_0 \cap K_1$ and $K_0 \cap K_2$ are nonempty closed (open) subcomplexes of K_0 and, since

$$K_0 = (K_0 \cap K_1) \cup (K_0 \cap K_2),$$

K_0 is not connected.

7.12. If K_0 is a connected subcomplex of a complex K and $T \in K$ is either a face of T_0 or has T_0 as a face, where $T_0 \in K_0$, then $K_0 \cup T$ is a connected complex.

In fact, if

$$K_0 \cup T = K_1 \cup K_2,$$

where K_1 and K_2 are closed (open) in $K_0 \cup T$, then, by 7.11, we may assume that $K_0 \subset K_1$. But then, since K_1 is closed (open) in $K_0 \cup T$, we have $T \in K_1$ so that K_2 is empty.

7.13. The union Q of an arbitrary number of connected subcomplexes K_α of a complex K containing a given fixed element $T \in K$ is connected.

Indeed, let

$$Q = Q_1 \cup Q_2$$

be a decomposition of Q into two subcomplexes closed in Q; let $T \in Q_1$; then, by 7.11, every K_α is contained in Q_1, i.e., $Q = Q_1$ and $Q_2 = 0$.

7.13 implies

7.14. The union of all the connected subcomplexes of a complex K containing a given $T \in K$ is a connected subcomplex $Q_K(T)$, the *component of T in K*.

A component $Q_K(T)$ obviously has the following *maximal property*: there does not exist any connected subcomplex $Q' \subset K$ different from $Q_K(T)$ and containing $Q_K(T)$.

7.13 further implies that there cannot exist two nonidentical disjoint connected subcomplexes Q_1 and Q_2 of K containing T and satisfying the condition of maximality (since $Q_1 \cup Q_2$ would be connected and $Q_1 \cup Q_2 \supset Q_1$, $Q_1 \cup Q_2 \supset Q_2$).

Therefore,

7.15. The component $Q_K(T)$ may be defined as the unique maximal connected subcomplex of K containing T.

Furthermore:

7.16. Two components $Q_K(T)$ and $Q_K(T')$ having a nonempty intersection are identical.

Finally, 7.12 implies

7.17. Every component $Q_K(T)$ is both closed and open in K.

Hence

7.1. *Every complex is uniquely partitioned into disjoint maximal connected subcomplexes, which are both open and closed. These subcomplexes are the components of K. A connected complex consists of one component.*

We note in conclusion that 7.12 implies

7.18. If $T < T'$, then $Q_K(T) = Q_K(T')$.

§7.2. The case of unrestricted simplicial complexes. Let K be an unrestricted simplicial complex. 7.18 implies that for an arbitrary $T \in K$ and arbitrary vertex $e < T$ the components $Q_K(T)$ and $Q_K(e)$ are identical. Hence we need consider only the components of the vertices of K. Furthermore, the definition of connectedness in the present case may be phrased as:

7.20. An unrestricted simplicial complex K is connected if every two nonempty closed subcomplexes K_1 and K_2 whose union is K have at least one common vertex.

Finally, let us call every finite sequence of edges (1-simplexes) of a complex K of the form

$$(e_1e_2),\ (e_2e_3),\ \cdots,\ (e_{s-1}e_s)$$

a *broken line joining the vertices* e_1 *and* e_s in the complex K.

7.21. *The component of a vertex e of an unrestricted simplicial complex K consists of all the simplexes of K whose vertices can be joined by a broken line to the vertex e.*

7.21 follows easily from:

7.22. *An unrestricted simplicial complex K is connected if, and only if, any two of its vertices can be connected by a broken line in K.*

Proof. If K can be represented in the form of a union of two closed subcomplexes K' and K'' having no common vertices, then no vertex of

K' can be joined by a broken line to any vertex of K'' [since the broken line would necessarily have a link $(e_i e_{i+1})$, where $e_i \in K'$, $e_{i+1} \in K''$, and consequently this link itself could not belong either to K' or K''].

This proves the first part of Theorem 7.22.

To prove the second part of the theorem, let us assume that e' and e'' are two vertices of K which cannot be joined by a broken line; let us consider the complex $K_{e'}$ consisting of all the elements of the complex K whose vertices can be joined to e' by a broken line. The complement of $K_{e'}$, the subcomplex $K'' = K \setminus K_{e'}$, is nonempty, since it contains the vertex e''. Both complexes $K_{e'}$ and K'' are closed and have no common vertices. At the same time $K = K_{e'} \cup K''$. This proves the theorem.

§7.3. The components of $\| K \|$ and K. Let K be a triangulation of a polyhedron $\| K \|$.

7.31. If K is a connected complex, $\| K \|$ is a connected polyhedron.

Indeed, let p and p' be two points of $\| K \|$, T and T' their carriers in K, e and e' any vertices of T and T', respectively. Let

$$(ee_1),\ (ee_2),\ \cdots,\ (e_{s-1}e_s),\ (e_s e')$$

be the links of a broken line joining e and e' in K; and let $[pe]$ and $[e'p']$ be straight line segments in $\overline{T}$ and $\overline{T}'$, respectively.

Then

$$[pee_1e_2 \cdots e_{s-1}e_s e' p']$$

is a broken line (in the elementary geometric sense) joining p and p' in $\| K \|$. This proves that $\| K \|$ is connected.

Again, it follows from 7.1, 3.1, and 3.22 that:

If $Q_1, \cdots, Q_s$ are components of a complex K, then the sets $\| Q_i \|$ are mutually disjoint polyhedra which are both closed and open in $\| K \|$.

It follows immediately that every component of a polyhedron $\| K \|$ is contained in some polyhedron $\| Q_i \|$ and, since $\| Q_i \|$ is connected, is identical with this $\| Q_i \|$.

Hence

7.3. *The components of a polyhedron $\| K \|$ coincide with the bodies of the components of any triangulation of $\| K \|$.*

Corollary 7.30. All the triangulations of a polyhedron consist of the same number of components, equal to the number of components of the polyhedron.

Chapter V

SPERNER'S LEMMA AND ITS COROLLARIES

All the results of this chapter are proved anew in Chapters X and XIV and again in Chapters X and XV, but in a less elementary fashion than here. This chapter can be read without reading the preceding chapters. It is merely necessary to consult the indicated references to Chapter I and to read the preliminary introductory section of Chapter IV on simplexes.

In what follows, to avoid ambiguity in the statement of theorems, we shall use the term "dimension number" to designate the combinatorial dimension of a simplex or complex, that is, the number of vertices of a simplex less 1 in the former case and the maximum of the dimension numbers of the simplexes of a complex in the latter case. In the same way, a polyhedron will be said to have dimension number n if it has a triangulation whose dimension number is n. The term "dimension" will be reserved for the topological concept defined in I, 8.42.

Several fundamental topological theorems, first established by Brouwer, are proved in this chapter. First among these is:

THEOREM I. *The dimension* (I, Def. 8.42) *of a closed simplex is equal to its dimension number* (i.e., to the number of its vertices less 1).

In other words (I, 8):

THEOREM I′ (THE PFLASTERSATZ FOR A SIMPLEX). *A closed n-simplex has closed ϵ-coverings of order $n + 1$ for every $\epsilon > 0$; if ϵ is sufficiently small, every closed ϵ-covering of a closed n-simplex is of order $\geq n + 1$.*

The first assertion of this theorem is contained in IV, 5.34 and 4.22. Essentially, this assertion is perfectly elementary and the reader may, without turning to Chapter IV, prove it as an exercise (in this connection see the hints given in §1).

The second assertion of Theorem I′ expresses a deep geometric fact; it also makes up the basic content of this chapter.

An immediate corollary of the second assertion of Theorem I′ is (the reader omitting Chapter IV should at once go on to Theorem III):

THEOREM II′. *Let Φ be a polyhedron of dimension number n. For every sufficiently small $\epsilon > 0$ every closed ϵ-covering of the polyhedron Φ has order $\geq n + 1$.*

This and IV, 5.34 imply

THEOREM II. *Every polyhedron Φ of dimension number n has dimension n.*

Furthermore, from Theorem I of this chapter we deduce

THEOREM III (INVARIANCE OF THE DIMENSION NUMBER OF R^n). *Two Euclidean spaces R^n and R^m of different dimension numbers n and m are not homeomorphic.*

The following proposition is an important and stronger form of Theorem III. It is proved in §3.

THEOREM IV (INVARIANCE OF INTERIOR POINTS OF SUBSETS OF R^n UNDER TOPOLOGICAL MAPPINGS INTO R^n). *A topological mapping C of a set $A \subseteq R^n$ onto a set $B \subseteq R^n$ maps every interior point of A (relative to R^n) into an interior point (relative to R^n) of B (and consequently maps a noninterior point of A into a noninterior point of B).*

REMARK. Theorem IV can be proved not only for the space R^n, but also for all topological manifolds M^n (see I, 5.3).

The proofs of the above theorems given in this chapter differ from the original proofs of Brouwer; they are based, essentially, on a single elementary proposition of combinatorial character known as Sperner's lemma (formulated and proved in §2). Sperner's lemma not only leads to surprisingly simple proofs of Theorems I–IV, but is in itself a remarkable geometric fact.

Sperner's lemma also enables us to give a very simple proof of yet another classical theorem of Brouwer, namely, the *fixed point theorem for an arbitrary continuous mapping of a closed simplex into itself* (or, of course, of an arbitrary compactum homeomorphic to a closed simplex). This proof of Brouwer's theorem is due to Knaster, Kuratowski, and Mazurkiewicz and has become classical. It is given in §4.

§1. Preliminary remarks

This section is intended only for the reader omitting Chapter IV.

§1.1. Triangulations and barycentric subdivisions of a closed simplex. Let us recall the propositions of Chapter IV required in this chapter.

a) We require only triangulations of a *closed simplex* $\overline{T}^n$. This term (triangulation of a closed simplex) denotes a finite set K^n of mutually disjoint simplexes (of various dimension numbers) satisfying the following conditions:

1. Every face of a simplex which is an element of K^n is itself an element of the set K^n.

2. Every element T^r of K^n is contained (as a point set) in one, and obviously in only one, of the faces (proper or not) of the simplex T^n, called the *carrier* of T^r.

3. The point set union of all the elements of K^n is the closed simplex $\overline{T}^n$.

This definition implies that:

Every face T^r of the simplex T^n is the union of the simplexes of the subdivision K^n of which T^r is the carrier.

Indeed, if $p \in T^r$ and $T^h{}_i \in K^n$ contains p, then the carrier of the simplex $T^h{}_i$ is necessarily T^r, since in the contrary case two faces of the simplex

T^n (namely, T^r and the carrier of the simplex $T^h{}_i$) would have a common point p. Hence every point $p \in T^r$ is contained in a simplex of K^n carried by T^r, which proves the assertion.

Furthermore,

If $T_j < T_i \in K^n$ and T^r is the carrier of T_i, then the carrier of T_j is a face of T^r.

Indeed, $\overline{T}^r$ is a closed set which is the union of all the faces of T^r. Therefore every contact point of T^r is contained in some face of T^r. Since T_j consists of contact points of T_i, and therefore also of T^r, the carrier of T_j is a face of T^r, which was to be proved.

b) Among all the triangulations of a closed simplex $\overline{T}^n$ the most important is its *barycentric subdivision*; the elements of the barycentric subdivision of a closed simplex $\overline{T}^n$ are all the faces of the $(n + 1)!$ simplexes of the form

$$(oo_{i_1}o_{i_1i_2} \cdots o_{i_1i_2\cdots i_n}),$$

where o is the centroid of the simplex T^n, o_{i_1} is the centroid of any of its $(n - 1)$-faces $T^{n-1}{}_{i_1}$, $o_{i_1i_2}$ the centroid of any $(n - 2)$-face $T^{n-2}{}_{i_1i_2}$ of $T^{n-1}{}_{i_1}$, etc., up to the vertex $o_{i_1\cdots i_n}$ of T^n.

The barycentric subdivision of a closed segment is obtained by dividing it in two. The barycentric subdivision of a triangle is shown in Fig. 95. One of the simplexes of the barycentric subdivision of a tetrahedron is shown in IV, Fig. 87.

It follows from 4.2 of Appendix 1 that:

If the diameter of a closed simplex $\overline{T}^n$ is equal to d, then the diameter of an arbitrary simplex of the barycentric subdivision of $\overline{T}^n$ does not exceed $[n/(n + 1)]\, d$.

Hence it follows that for sufficiently large h the barycentric subdivision of order h of $\overline{T}^n$ will yield a triangulation of this simplex of arbitrarily small mesh, i.e., a triangulation all of whose simplexes have diameters less than an arbitrarily given $\epsilon > 0$. (The mesh of a triangulation is the maximum of the diameters of its simplexes.)

c) We shall require the following fact: *A closed simplex $\overline{T}^n$ can be represented as the union of $n + 1$ closed sets $A_0, A_1, \cdots, A_n$, corresponding* (1–1) *to the vertices $e_0, e_1, \cdots, e_n$ of the simplex T^n in such a way that A_r contains the vertex e_r and has no point in common with the closed face of T^n opposite the vertex e_r.*

This fact is implicit in the results of Chapter IV: it suffices to define A_r as the closed barycentric star with center e_r, i.e., as the union of all the simplexes of the barycentric subdivision of $\overline{T}^n$ which have e_r as a vertex.

The required sets A_r can be constructed somewhat differently: let

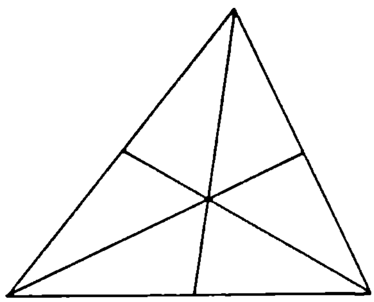


FIG. 95

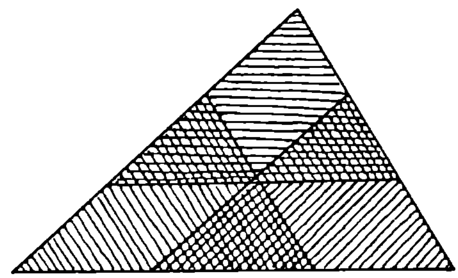


FIG. 96

R_r^{n-1} be the plane through the centroid of T^n parallel to the plane of the face $(e_0 \cdots e_{r-1}e_{r+1} \cdots e_n)$. This plane cuts off from T^n a closed simplex containing the vertex e_r which we also denote by A_r (Fig. 96).

d) *For arbitrary $\epsilon > 0$ the whole space R^n can be covered by a countable number of closed sets A_r of diameter $<\epsilon$ in such a way that no point of the space is contained in more than $n + 1$ of these sets and such that any sphere*

$$x_1^2 + x_2^2 + \cdots + x_n^2 \leq a^2$$

intersects only a finite number of the sets A_r.

To prove this, let us first construct a *simplicial ϵ-decomposition* ("infinite triangulation") K of the space R^n, i.e., a set K of mutually disjoint simplexes (of various dimensions) possessing the following properties:

1. Every face of an arbitrary simplex of the set K is itself an element of the set K.
2. The union of all the simplexes of K is R^n.
3. Every solid sphere of the space R^n intersects at most a finite number of simplexes of the set K.
4. The mesh of K is $< \epsilon$.

Such a set of simplexes (*infinite complex*) K can be constructed in the following way: let us divide R^n into the cubes

$$m_i \epsilon' < x_i < (m_i + 1) \epsilon' \qquad (i = 1, \cdots, n),$$

$$\epsilon' = \epsilon/2n^{\frac{1}{2}}, \qquad m_i = 0, \pm 1, \pm 2, \cdots,$$

and their faces. Let us divide each of these cubes barycentrically. This means: divide all their edges in half, then divide each square into eight equal triangles, projecting from the center of a square its boundary which has already been subdivided. Similarly, subdivide all the three-dimensional cubes, projecting from the center of each cube its already subdivided boundary, etc.

The resulting simplexes of various dimensions make up the required complex K.

Every point $p \in R^n$ is contained in one and only one simplex of K, the carrier of p.

Having constructed a simplicial ϵ-decomposition K of R^n, let us represent each of the closed n-simplexes of this decomposition in the form of a union of $n + 1$ closed sets A_i satisfying the conditions of c). If e_m is any vertex of K, denote by Φ_m the union of all the sets A_i just constructed which contain e_m. Since the simplexes of the decomposition have diameters $<\epsilon$, the sets Φ_m have diameters $<2\epsilon$. It is clear that an arbitrary bounded domain of R^n intersects only a finite number of the sets Φ_m. We shall prove, finally, that no point $p \in R^n$ is contained in more than $n + 1$ sets Φ_m. Indeed, from the construction of the sets Φ_m it follows that:

If the carrier of a point $p \in R^n$ is the simplex $T^r = (e_{h_0} \cdots e_{h_r}) \in K$, then p can be contained only in the sets $\Phi_{h_0}, \cdots, \Phi_{h_r}$.

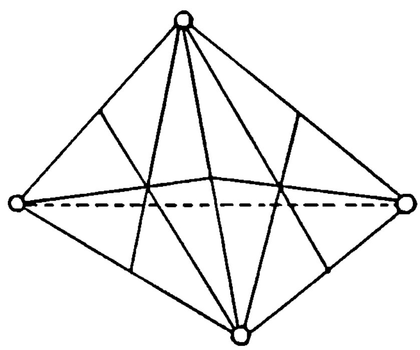


FIG. 97

It follows from what has just been proved (independently of the results of Chapter IV) that:

Every compactum in R^n (in particular, every closed simplex of R^n) *has a closed* ϵ*-covering of order* $\leq n + 1$ *for every* $\epsilon > 0$.

This proves the first assertion of Theorem I′.

e) In §3 we shall require the theorem asserting that the boundary of an n-simplex T^n has closed ϵ-coverings of order $\leq n$ for every $\epsilon > 0$. This proposition is also a special case of I, 5.34. However, it is also easy to prove directly.

To this end, it suffices to represent every closed $(n - 1)$-face of the simplex T^n as the union of sets A_i satisfying the conditions of c) and to define Φ_m for every vertex e_m of T^n as the union of the sets A_i containing the vertex e_m.

f) Let us note finally that the barycentric subdivisions of all the closed $(n - 1)$-faces of a given n-simplex T^n yield, by definition, the *barycentric subdivision of the boundary* $\overline{T}^n \setminus T^n$ *of* $\overline{T}^n$. Fig. 97 shows the case $n = 3$ (only the faces turned to the observer are shown).

§2. Sperner's lemma

§2.1. Sperner's lemma.

2.1. *Let* $T^n = (e_0 \cdots e_n)$ *be an* n*-simplex and let* K^n *be a triangulation of its closure* $\overline{T}^n$. *Let each vertex* $e'_k \in K^n$ *be made to correspond to a vertex* $Se'_k = e_{i_k}$ *of the simplex* T^n *in such a way that the following condition is satisfied:*

2.10. Se'_k *is a vertex of the carrier of* e'_k *(i.e., a vertex of that face of* T^n *which contains* e'_k*).*

Then there exists an n*-simplex*

$$T^n{}_i = (e'_{k_0} \cdots e'_{k_n})$$

of the triangulation K^n *such that all the vertices* $Se'_{k_0}, \cdots, Se'_{k_n}$ *are distinct.*

Proof. Let $T^n{}_1, \cdots, T^n{}_s$ be all the n-simplexes of the triangulation K^n. Let us call a simplex $T^n{}_i$ normal, if all its vertices have been made to correspond to distinct vertices of T^n. We shall prove the following proposition which is stronger than Sperner's lemma:

2.11. *The number of normal simplexes is odd.*

Theorem 2.11 is obvious for $n = 0$. Let us suppose that it has been proved for all $(n - 1)$-simplexes and prove it for an n-simplex T^n.

Let us say that an $(n - 1)$-face of a simplex $T^n{}_i$ is "marked" if its vertices have been made to correspond to the vertices $e_1, e_2, \cdots, e_n$ of T^n.

We note first that:

2.12. *The number of marked faces of a simplex* $T^n{}_i$ *is either* 1, *or* 2, *or* 0, *while the number of marked faces of* $T^n{}_i$ *is equal to* 1 *if, and only if,* $T^n{}_i$ *is normal.*

Indeed, if a simplex $T^n{}_i = (e'_{i_0} \cdots e'_{i_n})$ is normal and

$$Se'_{i_0} = e_0, \quad Se'_{i_1} = e_1, \cdots, Se'_{i_n} = e_n,$$

then $(e'_{i_1} \cdots e'_{i_n})$ is obviously the unique marked face of $T^n{}_i$.

Suppose the simplex $T^n{}_i$ is not normal but has, nevertheless, a marked face $(e'_{i_1} \cdots e'_{i_n})$, so that, for instance, $Se'_{i_1} = e_1$, $Se'_{i_2} = e_2$, $\cdots$, $Se'_{i_n} = e_n$. Then Se'_{i_0} is one of the vertices $e_1, \cdots, e_n$, say $Se'_{i_0} = e_1$, and the simplex $T^n{}_i$ has exactly two marked faces, namely,

$$(e'_{i_1} \cdots e'_{i_n}) \quad \text{and} \quad (e'_{i_0}e'_{i_1} \cdots e'_{i_n}).$$

This proves 2.12.

Let us denote by a_i the number of marked faces of a simplex $T^n{}_i$, and let us set $a = \sum_{i=1}^{s} a_i$.

It follows from what has just been proved that the number of normal simplexes has the same parity as the number a. Therefore, it suffices to prove that the number a is odd.

Let us consider any $(n - 1)$-simplex

$$T_j^{n-1} \in K^n.$$

Three cases are possible.

1. *T_j^{n-1} is contained in the interior of the simplex T^n.* Then either T_j^{n-1} is not a marked face of any simplex T^n_i or the simplex T_j^{n-1} is a marked face of exactly two simplexes T^n_i and T^n_h, which have T_j^{n-1} as their common face. In the latter case T_j^{n-1} will be counted twice in the calculation of the sum $a = \sum_{i=1}^{s} a_i$.

2. *The simplex T_j^{n-1} is contained in a face of T^n* distinct from the face $T^{n-1} = (e_1 \cdots e_n)$.

In this case, by virtue of 2.10, the simplex T_j^{n-1} cannot be a marked face of any simplex T^n_i.

3. *The simplex T_j^{n-1} is contained in the face $T^{n-1} = (e_1 \cdots e_n)$.*

This case we divide anew into two cases:

3a. At least two vertices of T_j^{n-1} have been mapped into the same vertex of $T^{n-1} = (e_1 \cdots e_n)$. Then T_j^{n-1} cannot be a marked face of any simplex T^n_i.

3b. All the vertices of T_j^{n-1} have been mapped into distinct vertices of $T^{n-1} = (e_1 \cdots e_n)$. Hence T_j^{n-1} is a marked face of a unique simplex T^n_i

From this classification of the different cases it follows that the parity of the number a is the same as the parity of the number of all $(n - 1)$-simplexes T_j^{n-1} which satisfy 3b.

Let us consider case 3b more closely. The simplexes of K^n contained in T^{n-1} and its faces form a triangulation K^{n-1} of the closed simplex $\overline{T}^{n-1}$, where every vertex of K^{n-1} has been mapped by S into one of the vertices of T^{n-1} so as to satisfy the conditions of Theorem 2.1 for the dimension $(n - 1)$. Consequently, by the inductive hypothesis, the number of those $(n - 1)$-simplexes contained in T^{n-1}, all of whose vertices have been mapped into distinct vertices of T^{n-1}, is odd. In other words, the number of simplexes T_j^{n-1} which satisfy 3b is odd, and this means that a is also odd. This completes the proof of 2.1.

§2.2. Corollary of Sperner's lemma. Conclusion of the proof of the Pflastersatz. From Sperner's lemma we deduce the corollary:

2.21. *Let T^n be an n-simplex with vertices $e_0, e_1, \cdots, e_n$. Let*

$$\alpha = \{A_0, \cdots, A_n\}$$

be a covering of the closed simplex $\overline{T}^n$, consisting of $n + 1$ closed sets $A_0, \cdots, A_n$ satisfying the condition: every closed face $[e_{i_0} \cdots e_{i_r}]$ of $\overline{T}^n$ is contained in $A_{i_0} \cup \cdots \cup A_{i_r}$ (in particular $e_i \in A_i$ for $i = 0, 1, \cdots, n$).

Then the intersection of all the sets A_i is nonempty.

Proof. Obviously (on the basis of Lebesgue's lemma, I, 8.32) 2.21 follows from:

2.22. If K^n is a triangulation of the closed simplex $\overline{T}^n$ of arbitrarily small mesh, there is at least one closed simplex $\overline{T}^n{}_1$ of K^n which intersects all of the sets $A_0, \cdots, A_n$.

To prove Theorem 2.22, let us assign to each vertex $e'_j \in K^n$ a vertex $Se'_j = e_i$ of the carrier of e'_j such that

$$e'_j \in A_i.$$

Obviously, the mapping S satisfies condition 2.10 of Sperner's lemma.

Applying Sperner's lemma to the mapping S, we find a simplex $T^n{}_1 \in K^n$ all of whose vertices have been mapped into distinct vertices of T^n. But this means that the skeleton of $T^n{}_1$ intersects all of the sets $A_0, \cdots, A_n$, whence 2.22 follows.

From Theorem 2.21 we easily deduce:

2.23. *Let*

$$\alpha = \{A_0, \cdots, A_n\}$$

be a closed covering of a closed simplex $\overline{T}^n = [e_0 \cdots e_n]$ *such that*

1. $e_i \in A_i$,

2. A_i *has no points in common with the closed face* $\overline{T}_i^{n-1}$ *opposite the vertex* e_i.

Then

$$A_0 \cap \cdots \cap A_n \neq 0.$$

To prove 2.23, it suffices to show that the hypothesis of Theorem 2.23 implies the hypothesis of Theorem 2.21. Let $[e_{i_0} \cdots e_{i_0}]$ be a closed face of T^n and let i be different from $i_0, \cdots, i_r$. Then $[e_{i_0} \cdots e_{i_r}] \subseteq \overline{T}_i^{n-1}$, so that A_i does not meet $[e_{i_0} \cdots e_{i_r}]$. Since this is valid for arbitrary $i \neq i_0, \cdots, i_r$, $[e_{i_0} \cdots e_{i_r}] \subseteq A_{i_0} \cup \cdots \cup A_{i_r}$.

Let us now pass to the proof of the second assertion of the Pflastersatz:

2.24. *For sufficiently small* $\epsilon > 0$ *every closed* ϵ*-covering of a closed* n*-simplex* $\overline{T}^n$ *has order* $\geq n + 1$.

Proof. Let ϵ be so small that no set of diameter less than ϵ meets all the closed $(n - 1)$-faces of T^n [since the intersection of all the closed $(n - 1)$-faces of T^n is empty, such an ϵ can be found by Lebesgue's lemma, I, 8.32].

With ϵ as above, let

$$\alpha = \{A_0, \cdots, A_s\}$$

be any ϵ-covering of $\overline{T}^n$. In virtue of the choice of ϵ, no A_j containing a given vertex e_i of T^n intersects the closed face $\overline{T}_i^{n-1}$ opposite the vertex

e_i. Hence, in particular, it follows that no A_i contains more than one vertex of T^n.

Let us now number the elements of the covering so that

$$e_0 \in A_0, \cdots, e_n \in A_n.$$

If $s > n$, let A_j be any set with index $j > n$, and let $\overline{T}_i^{n-1}$ be the closed face which does not meet A_j. Let us delete the element A_j from the covering α and replace A_i by $A_i \cup A_j$, denoting this last set again by A_i. After this replacement the covering α is converted into a covering $\alpha^{(1)}$, which has one element less than α. The order of $\alpha^{(1)}$, as easily seen, is in any case no greater than the order of α. It is also easily shown that no element of the covering $\alpha^{(1)}$ containing a vertex of T^n intersects the closed face opposite this vertex.

Repeating this operation as many times as necessary, we finally obtain a covering

$$\alpha^{(p)} = \{A_0, \cdots, A_n\}$$

satisfying all the conditions of Theorem 2.23. Consequently, there exists a point contained in all the sets $A_0, \cdots, A_n$, i.e., the order of the covering $\alpha^{(p)}$ is not less than $n + 1$. But since the above replacement of one covering by another did not increase the order of the covering, the original covering α had order $\geq n + 1$. This proves the theorem.

§2.3. The invariance of the dimension number of R^n.

2.31. *Every compactum $\Phi \subset R^n$ has dimension $\leq n$.*

Indeed, every compactum $\Phi \subset R^n$, being a bounded set, is contained in some closed n-simplex $\overline{T}^n$; since $\overline{T}^n$ has closed ϵ-coverings of order $n + 1$ for every $\epsilon > 0$, Φ also has closed ϵ-coverings of order $n + 1$ for every ϵ.

Hence:

2.32. *No set $A \subseteq R^n$ containing an interior point* (relative to R^n) *can be mapped topologically into R^m, where $m < n$.*

Indeed, such a mapping would take an n-simplex contained, by assumption, in A, into some compactum of dimension $<n$. This is impossible, since the dimension of a compactum is invariant under topological mappings.

In particular, R^n itself cannot be mapped topologically into R^m for $m < n$. Hence it follows that:

2.33. Invariance of the Dimension Number of R^n. *For $n \neq m$, the spaces R^n and R^m are not homeomorphic.*

From 2.33 and the definition of a topological manifold (I, 5.3) it follows that:

2.34. *If $m \neq n$, an n-manifold cannot be homeomorphic to an m-manifold.*

§3. Invariance of interior points

§3.1. *A topological mapping of a set $A \subseteq R^n$ onto a set $B \subseteq R^n$ maps every interior point of A into an interior point of B, and every noninterior point of A into a noninterior point of B.*

Here "interior point" ("noninterior point") means "interior point" ("noninterior point") *relative to R^n.*

The proof is based on the following lemma:

LEMMA 3.11. *Let*

$$\alpha = \{A_1, \cdots, A_s\}$$

be a closed covering of order $n + 1$ of a closed bounded set $\Phi \subset R^n$ with the property that there is exactly one point $p \in \Phi$ contained in $n + 1$ elements of α. Then if p is not an interior point of Φ, the covering α can be transformed into a covering α' of order $\leq n$ by a modification of α in an arbitrarily small neighborhood of p.

[A modification of a covering α in a neighborhood O_p of p is a transition from $\alpha = \{A_1, \cdots, A_s\}$ to a closed covering $\alpha' = \{A'_1, \cdots, A'_s\}$ such that $A_i \setminus O_p = A'_i \setminus O_p$, $i = 1, \cdots, s$.]

Proof of the lemma. Let T^n be an arbitrarily small n-simplex containing p in its interior. Every point of the boundary $\| \dot{T}^n \|$ of T^n is contained in no more than n elements of α. Therefore, there exists an $\epsilon > 0$ such that the ϵ-neighborhood of every point of $\| \dot{T}^n \|$ intersects no more than n elements of α.

Let us now take any closed ϵ-covering

$$\beta = \{B_1, \cdots, B_m\}$$

of order n of the set $\| \dot{T}^n \|$.

Let us assign to every set $B_j \in \beta$ a set $A_{i(j)} \in \alpha$ in the following way:

1. If B_j does not intersect Φ, let $A_{i(j)}$ be any set A_i which meets $\overline{T}^n$.

2. If B_j meets Φ, then, because of the choice of ϵ, it meets no more than n elements of the covering α; let us denote any one of the elements of α which meets B_j by $A_{i(j)}$. In both cases the same set A_i may, of course, turn out to be the set $A_{i(j)}$ for different B_j's.

Let us now construct a system α'' of closed sets A_i'' as follows:

Every $A_i \in \alpha$ which is not a set $A_{i(j)}$ for any B_j is, by definition, an element A_i'' of the system α''; but if A_i is a set $A_{i(j)}$ for one or several B_j's, then define A_i'' as the union of all such B_j's and the set A_i. Obviously

$$\Phi \cup \| \dot{T}^n \| \subseteq \bigcup A_i'', \qquad A_i'' \in \alpha''.$$

The systems α'' and α are in (1–1) correspondence, where, for each i, either $A_i'' = A_i$ or A_i'' is the union of the set A_i and certain subsets of

the set $\| \dot{T}^n \|$. Therefore, if a point q is contained in the sets $A''_{i_1}, \cdots, A''_{i_r}$, q is contained in either the sets $A_{i_1}, \cdots, A_{i_r}$ or in $\| \dot{T}^n \|$ (these cases, of course, are not mutually exclusive). Therefore to show that p is the unique point contained in $n + 1$ elements of α'', it suffices to prove that no point of $\| \dot{T}^n \|$ is contained in more than n elements of α''.

Let us prove that no point of $\| \dot{T}^n \|$ is contained in more than n elements of the system α''.

Let $q \in \| \dot{T}^n \|$. If $q \notin \Phi$, then q is contained in no more than n sets B_j, say $B_{j_1}, \cdots, B_{j_m}$; the number of corresponding $A_{i(j)}$'s, say $A_{i_1}, \cdots, A_{i_m}$, will not exceed n; the point q is contained only in the elements $A''_{i_1}, \cdots, A''_{i_m}$ of α'', i.e., in no more than n elements.

Now let $q \in \| \dot{T}^n \| \cap \Phi$ and let $A_1, \cdots, A_h, B_1, \cdots, B_k$ be all the elements of the coverings α and β containing q. All the sets $B_1, \cdots, B_k$ are contained in an ϵ-neighborhood $O(q, \epsilon)$ of q, and certainly among all the sets $A_1, \cdots, A_h$ there will be no more than n elements of the covering α meeting $O(q, \epsilon)$. Thus let

$$A_1, \cdots, A_h, \cdots, A_g, \qquad g \leq n,$$

be all the elements of α meeting $O(q, \epsilon)$. All the $A_{i(j)}$, $j = 1, 2, \cdots, k$, are among these sets. Therefore $A_1'', \cdots, A_g''$ are the only elements of α'' which can contain q. Since $g \leq n$, the assertion is proved.

Let us now take in the interior of the simplex T^n any point o not contained in Φ (since p is not an interior point of Φ, the required point o can always be found).

Let us consider those elements of the system α'' which meet $\| \dot{T}^n \|$; let these be

$$A_1'', \cdots, A_v''.$$

Denoting by $[oA_i'']$ the union of all closed segments of the form $[oq]$, where $q \in \| \dot{T}^n \| \cap A_i''$, set

$$A_i''' = [A_i'' \setminus (A_i'' \cap T^n)] \cup [oA_i''], \qquad i = 1, \cdots, v,$$

and

$$A_i''' = A_i'' \setminus T^n, \qquad i > v.$$

The set A_i''' can differ from A_i'' only in points of T^n. Therefore a point not in $\overline{T}^n$ cannot be contained in more than n sets A_i'''. As for the points of $\overline{T}^n$, only the point o can be contained in $n + 1$ sets A_i''' (since, in the contrary case, if o' were such a point, the endpoint q of $[oo'q]$ on $\| \dot{T}^n \|$ would belong to $n + 1$ sets A_i'', in contradiction to what was proved above). Since $o \notin \Phi$, setting

$$A'_i = A_i''' \cap \Phi \qquad \text{for all } i,$$

we obtain a covering

$$\alpha' = \{A'_1, \cdots, A'_s\}$$

of the set Φ of order $\leq n$.

Obviously, every A'_i can differ from the corresponding A_i only in points belonging to $\overline{T}^n$, but since the simplex T^n can be taken arbitrarily small, we are justified in saying that the covering α' was obtained from the covering α by a modification of the latter in an arbitrarily small neighborhood of p. Hence the lemma is proved.

From the lemma we deduce

3.12. *A topological mapping C of a closed n-simplex $\overline{T}^n$ into the Euclidean n-space R^n maps every interior point p of T^n into an interior point of the set $\Phi = C(\overline{T}^n)$.*

Proof. Let $T^n = (e_0 \cdots e_n)$. Take the barycentric subdivision of the boundary of T^n and project it from the point p. The resulting triangulation K of the closed simplex $\overline{T}^n$ differs from a barycentric subdivision only in that the "centroid" of the simplex T^n is the point p. Let us denote by A_i the union of the closures of all the simplexes of K having e_i as a vertex.

The point p is the unique common point of the sets $A_0, \cdots, A_n$; here both the covering

$$\alpha = \{A_0, \cdots, A_n\}$$

of the closed simplex $\overline{T}^n$ and every covering which is obtained from α by a modification of the sets A_i in a sufficiently small neighborhood of p satisfies 2.23.

Hence it follows that:

For a sufficiently small $\epsilon > 0$ every covering α' obtained from α by a modification of the sets A_i in an ϵ-neighborhood of p is a covering of order $\geq n + 1$.

Now let $q = C(p)$ be a boundary point of the set $\Phi = C(\overline{T}^n)$. Take $\delta > 0$ so small that the image of a δ-neighborhood of q (relative to Φ) under the mapping C^{-1} is contained in an ϵ-neighborhood of p. Applying Lemma 3.11, modify the sets $C(A_i)$ in the δ-neighborhood of q in such a way that the result is a covering β' of order $\leq n$ of the set Φ. C^{-1} maps the covering β' into a covering α' of the same order $\leq n$, which is a modification of α in an ϵ-neighborhood of p. This is impossible, and 3.12 follows.

We can now prove Theorem 3.1 in a few words. Let p be an interior point of the set $A \subseteq R^n$; p is an interior point of some closed n-simplex $\overline{T}^n \subseteq A$ and, therefore, the mapping C of $A \subseteq R^n$ takes p into an interior point q of the set $C(\overline{T}^n)$ and hence into an interior point of the set $C(A)$.

C maps every noninterior point p of A into a noninterior point $C(p)$ of

$C(A)$, since, in the contrary, case C^{-1} would map an interior point $C(p)$ of $C(A)$ into a noninterior point p of A.

This completes the proof of Theorem 3.1.

Theorem 3.1 immediately implies the following proposition, which is a generalization in a merely formal sense, but which is required in the sequel.

3.13. *Let U^n and V^n be topological spaces homeomorphic to R^n; let C be a topological mapping of a set $A \subseteq U^n$ onto a set $B \subseteq V^n$; then C maps every interior point of A relative to U^n into an interior point of B relative to V^n.*

The following proposition is implicit in 3.13:

3.14. *Let U^n and V^n be homeomorphic to R^n; let C be a topological mapping of a set A open in U^n into the space V^n. Then $C(A)$ is open in V^n. In particular, every topological image of the whole space U^n in V^n is an open set in V^n.*

Since R^m is nowhere dense in R^n for $m < n$, a topological mapping of R^n onto R^m is impossible. We have therefore obtained another proof of Theorem 2.33.

REMARK. We note separately a special case of Theorem 3.14, which (for $n = 2$) was used in Chapter III.

3.140. *Let U^n and V^n be two sets* (contained in some topological space) *homeomorphic to R^n. If $U^n \subseteq V^n$, then U^n is open in V^n.*

DEFINITION 3.15. A compactum homeomorphic to a closed n-simplex is called a *closed n-cell.*

Since dimension is topologically invariant and the dimension of a simplex is equal to its dimension number, the number n in Def. 3.15 is really the dimension of an n-cell and is therefore unique (i.e., no topological space can be both an n-cell and an m-cell for $n \neq m$).

We shall prove the following important proposition:

3.16. *Every topological mapping of a closed n-simplex $\overline{T}^n$ onto a closed n-cell $\bar{E}^n$ takes the boundary $\overline{T}^n \setminus T^n$ of $\overline{T}^n$ onto a unique set S^{n-1}, called the boundary of the closed n-cell $\bar{E}^n$; the open set $\bar{E}^n \setminus S^{n-1}$ is dense in $\bar{E}^n$ and is denoted by E^n; it is called the interior of $\bar{E}^n$, or an open n-cell.*

Proof. Let C_1 and C_2 be two topological mappings of $\overline{T}^n$ onto $\bar{E}^n$. If, e.g., it were true for a point $x \in \overline{T}^n \setminus T^n$ that

$$C_2(x) \notin C_1(\overline{T}^n \setminus T^n),$$

then it would be true that $C^{-1}_1 C_2(x) \notin \overline{T}^n \setminus T^n$; however, $C^{-1}_1 C_2$ is a topological mapping of $\overline{T}^n$ onto itself, so that, by 3.1, $C^{-1}_1 C_2$ maps $\overline{T}^n \setminus T^n$ onto itself. The resulting contradiction proves that

$$C_2(\overline{T}^n \setminus T^n) \subseteq C_1(\overline{T}^n \setminus \overline{T}^n).$$

In the same way,

$$C_1(\overline{T}^n \setminus T^n) \subseteq C_2(\overline{T}^n \setminus T^n),$$

i.e.,

$$C_1(\overline{T}^n \setminus T^n) = C_2(\overline{T}^n \setminus T^n).$$

Since T^n is open and dense in $\overline{T}^n$, the set E^n, being the image of T^n under a topological mapping of $\overline{T}^n$ onto $\bar{E}^n$, is open and dense in $\bar{E}^n$.

This proves Theorem 3.16.

§3.2. Invariance of interior points of topological manifolds. The following very important theorem, a generalization of Theorem 3.1, may be easily derived from 3.1:

3.2. *A topological mapping of a subset A of a topological n-manifold $M^n{}_1$ onto a subset B of a topological n-manifold $M^n{}_2$ takes every interior point of A (relative to $M^n{}_1$) into an interior point of B (relative to $M^n{}_2$), every noninterior point of A (relative to $M^n{}_1$) into a noninterior point of B (relative to $M^n{}_2$).*

Proof. We may restrict ourselves to the proof of the assertion for interior points. After that, the proof for noninterior points is a repetition, word for word, of the last lines of the proof of Theorem 3.1.

Hence, let C be a topological mapping of $A \subseteq M^n{}_1$ onto $B \subseteq M^n{}_2$.

Let p be an interior point of A relative to $M^n{}_1$. Let V^n be a neighborhood of $C(p)$ in $M^n{}_2$ homeomorphic to R^n.

Let U^n be a neighborhood of p in A homeomorphic to R^n, such that $C(U^n) \subseteq V^n$.

Since U^n and V^n are homeomorphic to R^n, $C(U^n)$ is open in V^n; but V^n is open in $M^n{}_2$, so that $C(U^n)$ is open in $M^n{}_2$. Since $p \in U^n \subseteq A$, $C(p) \in C(U^n) \subseteq B$; this completes the proof.

§4. Fixed point theorem for an n-cell

We shall prove the following theorem:

FIXED POINT THEOREM FOR AN n-CELL. *If C is a continuous mapping of a closed n-cell $\bar{E}^n$ into itself, C has at least one fixed point* [i.e., there is a point $p \in \bar{E}^n$ such that $C(p) = p$].

We may obviously suppose, without loss of generality, that $\bar{E}^n$ is a closed n-simplex

$$\overline{T}^n = (e_0 \cdots e_n).$$

Let C be a continuous mapping of $\overline{T}^n$ into itself and let C map the point $p \in \overline{T}^n$ with barycentric coordinates $m_0, \cdots, m_n$ into the point $p' = C(p)$ with barycentric coordinates $m'_0, \cdots, m'_n$ (the barycentric coordinates are with respect to the system $\{e_0, \cdots, e_n\}$).

Denote by A_i the set of all those points $p \in \bar{T}^n$ for which $m'_i \leq m_i$. It is clear that every set A_i is closed. We shall prove that the sets A_i form a covering of $\bar{T}^n$ satisfying the hypothesis of Theorem 2.21, i.e., that an arbitrary closed face $[e_{i_0} \cdots e_{i_r}]$ of T^n is covered by the sets $A_{i_0}, \cdots, A_{i_r}$. Let p be an arbitrary point of the closed face $[e_{i_0} \cdots e_{i_r}]$. Then $m_{i_0} + \cdots + m_{i_r} = 1 \geq m'_{i_0} + \cdots + m'_{i_r}$, whence it follows that $m'_{i_k} \leq m_{i_k}$ for at least one i_k. Therefore $p \in A_{i_k}$.

Hence Theorem 2.21 can be applied to the sets A_i to yield a point p contained in every set A_i.

For this point p

$$m'_0 \leq m_0, \cdots, m'_n \leq m_n.$$

Hence, in virtue of the conditions

$$m'_0 + \cdots + m'_n = m_0 + \cdots + m_n = 1, \qquad 0 \leq m'_i,$$

it follows that

$$m'_0 = m_0, \cdots, m'_n = m_n.$$

In other words, $C(p) = p$, q.e.d.

Chapter VI

INTRODUCTION TO DIMENSION THEORY

All the concepts introduced in I, 8 are presupposed in this chapter.

The definition of dimension was given in I, 8.4. The basic theorem without which the very concept of dimension would have no right to existence, namely Brouwer's theorem that the dimension of a closed simplex is equal to its dimension number, was proved in Chapter V. It is easy to deduce from this theorem (V, Preliminary Remarks) that the dimension of a polyhedron is equal to the dimension number of an arbitrary triangulation of the polyhedron. This to some extent justifies the general concept of dimension. However, only to some extent; the introduction of a new fundamental, and not merely serviceable concept, is justified only if it substantially clarifies some sufficiently important branch of mathematics, and does not serve merely to prove one or two theorems. The concept of dimension fully justifies itself in this respect: the dimension theory developed in the last quarter of a century represents an integral part, geometric in content, of general topology. It has provided a deeper insight into various geometric properties, not only of compacta, but to some extent of their arbitrary subsets as well.

The full development of dimension theory is achieved only in the so called homology dimension theory, which is not presented in this book: it is based on certain theorems of the combinatorial topology of polyhedra which are not adduced here. We shall therefore be compelled to restrict ourselves to the first fundamental theorems of dimension theory, i.e., to give an introduction to this theory and refer the reader to deeper studies of the modern theory in the journal literature. (In addition to the classical papers of Brouwer [a] and Urysohn [a], and Menger's book [M], see Aleksandrov [b, d]. A systematic account of dimension theory, with an introduction to the geometric theory, is to be found in Hurewicz-Wallman [H-W].)

§1. Theorems on ϵ-displacements and imbedding in R^n

§1.1. Definition of ϵ-mappings and ϵ-displacements. Outline of the section.

Definition 1.11. A continuous mapping C of a compactum X into a compactum Y is called an *ϵ-mapping* if the inverse image $C^{-1}(y) \subseteq X$ of every point y of Y has diameter $<\epsilon$.

Definition 1.12. Let the compactum X be a subset of a metric space R. A continuous mapping C of X into R is called an *ϵ-displacement of the*

compactum X in the space R if $\rho(x, Cx) < \epsilon$ for every $x \in X$. If an ϵ-displacement C maps two points x and x' of X onto the same point y, then

$$\rho(x, x') \leq \rho(x, y) + \rho(y, x') < \epsilon + \epsilon = 2\epsilon.$$

Hence the diameter of $C^{-1}(y)$, equal (in view of the compactness of X) to the maximum of $\rho(x, x')$ for $x \in C^{-1}(y)$, $x' \in C^{-1}(y)$, is less than 2ϵ. Hence

1.13. *An ϵ-displacement C of a compactum X is a 2ϵ-mapping of X onto $C(X)$.*

EXAMPLE. Let X be the Hilbert parallelotope (I, 2.3). Let us assign to every point

$$x = (x_1, \cdots, x_n, \cdots) \in X$$

the point

$$C(x) = (x_1, \cdots, x_n, 0, 0, 0, \cdots) \in X.$$

The resulting mapping C of the Hilbert parallelotope onto an n-dimensional rectangular parallelotope is a $(1/2^{n-1})$-displacement. Therefore:

1.140. For every $\epsilon > 0$ there is an ϵ-displacement of the Hilbert parallelotope onto a polyhedron, namely a finite-dimensional parallelotope.

Hence:

1.141. Every closed subset of the Hilbert parallelotope can be mapped into a polyhedron by means of an ϵ-displacement for every $\epsilon > 0$.

Since all compacta are homeomorphic to subsets of the Hilbert parallelotope, it follows easily from 1.141 that:

1.142. *A compactum can be ϵ-mapped into a polyhedron for every $\epsilon > 0$.*

EXERCISE. Prove that a closed subset of Hilbert space can be mapped by an ϵ-displacement for every $\epsilon > 0$ into a polyhedron (and in general into a closed bounded set of a finite-dimensional Euclidean subspace of Hilbert space) only if it is a compactum. Prove the analogous proposition for ϵ-mappings (instead of ϵ-displacements).

OUTLINE OF THE FIRST SECTION. The first fundamental purpose of this section is to prove the following proposition:

THEOREM 1.4 (Theorem on ϵ-mappings). *Let Φ be a compactum. For every $\epsilon > 0$, Φ can be ϵ-mapped onto a polyhedron; in this connection, if* $\dim \Phi = n$, *then Φ can be ϵ-mapped for every ϵ onto an n-dimensional polyhedron and, for sufficiently small $\epsilon > 0$, cannot be ϵ-mapped into any polyhedron of dimension $<n$. If the compactum Φ is a subset of Hilbert space or of a Euclidean space, these ϵ-mappings can be realized by ϵ-displacements for every ϵ.*

The proof will proceed in the following order:

First we shall prove that for every n-dimensional compactum Φ there

exists an $\epsilon = \epsilon(\Phi)$ such that every compactum which is the image of Φ under any ϵ-mapping has dimension $\geq n$ (Theorem 1.16). The proof of this theorem is based on Lemma 1.15.

Next, we shall prove that the existence of an ϵ-mapping of the compactum Φ into a polyhedron Π implies the existence of an ϵ-mapping of Φ *onto* a polyhedron $\Pi_0 \subseteq \Pi$.

This transition from mappings *into* a polydedron to mappings *onto* a polyhedron is made very simply by means of a retraction (1.2). Along the way we shall prove the important Theorem 1.25 characterizing n-dimensional compact subsets of R^n.

The central part of the proof consists in constructing for every $\epsilon > 0$ an ϵ-mapping of the given n-dimensional compactum into an n-dimensional polyhedron. This is done in 1.3 and 1.4 by means of the so called *barycentric mappings.*

Finally, it remains to be proved that if Φ is a subset of Hilbert space or of Euclidean space R^m, the ϵ-mapping is realized by an ϵ-displacement. This is easily achieved in 1.5 (since the proof is based on IV, 3.1, Remark 2, and consequently makes use of properties of convex polyhedral domains not proved in this book, we accordingly give several weakened formulations independent of propositions not proved here). The method of proof consists in constructing for every continuous mapping C of the compactum Φ into the given space R^m a "barycentric mapping C_ω" (see above) into a polyhedron of the same dimension as Φ, differing from C by as little as desired.

If C is the identity mapping, C_ω is an ϵ-displacement.

These barycentric approximations to continuous mappings also lead to the solution of the second basic problem of this section, namely to the proof of Theorem 1.6, which asserts that every n-dimensional compactum is homeomorphic to a (closed and bounded) subset of R^{2n+1}: it turns out that the barycentric mappings approximating a given continuous mapping of an n-dimensional compactum Φ into R^m, where $m \geq 2n + 1$, sufficiently well, are ϵ-mappings. Hence for every ϵ the set of all ϵ-mappings of an n-dimensional compactum Φ into an m-dimensional closed solid sphere $\bar{E}^m$, $m \geq 2n + 1$, is dense in the space $\mathfrak{C}(\Phi, \bar{E}^m)$ of all continuous mappings of Φ into $\bar{E}^m$.

Since $\mathfrak{C}(\Phi, \bar{E}^m)$ is a complete metric space and the ϵ-mappings form an open set in $\mathfrak{C}(\Phi, \bar{E}^m)$, it follows easily that topological mappings of Φ into $\bar{E}^m$ not only exist but even form a set dense in $\mathfrak{C}(\Phi, \bar{E}^m)$. This will prove Theorem 1.6 and the stronger Theorem 1.63.

LEMMA 1.15. *Let C be an ϵ-mapping of a compactum X into a compactum Y. There exists a positive number η such that every set $B \subseteq Y$ of diameter $\leq \eta$ has an inverse image $C^{-1}(B)$ of diameter $< \epsilon$.*

Proof by contradiction. Suppose such an η does not exist. Then for every natural number n there is a set B_n in Y of diameter $<1/n$ whose inverse image $A_n = C^{-1}(B_n)$ has diameter $\geq \epsilon$.

Let x_n and x'_n be points of $\bar{A}_n$ for which

$$\rho(x_n , x'_n) = \delta(A_n) \geq \epsilon.$$

Passing, if necessary, to subsequences, we may suppose that the sequences $\{x_n\}$ and $\{x'_n\}$ converge to points $x \in X$ and $x' \in X$, respectively. Since $\rho(Cx_n , Cx'_n) < 1/n$, $Cx = Cx' = y$; on the other hand, $\rho(x, x') \geq \epsilon$, which contradicts the fact that C is an ϵ-mapping.

Let us apply Lemma 1.15 to the proof of the following important proposition:

1.16. *Let the dimension of a compactum Φ be equal to n. There exists an $\epsilon = \epsilon(\Phi) > 0$ such that every compactum Φ' which is the image of Φ under an ϵ-mapping has dimension $\geq n$.*

Proof. Choose an ϵ such that the compactum Φ has no closed ϵ-covering of order $\leq n$. Let

$$\Phi' = C(\Phi),$$

where C is an ϵ-mapping. Let us define a number η for this ϵ-mapping as in 1.15., and take any closed η-covering

$$\alpha' = \{A'_1 , \cdots , A'_s\}$$

of the compactum Φ'. Then the sets $A_i = C^{-1}(A'_i)$ form an ϵ-covering α of the compactum Φ whose order is equal to the order of the covering α'.

By the definition of ϵ, the order of the covering α is in any case $\geq n + 1$; the same is true for the order of the covering α'. Hence every η-covering of the compactum Φ' has order $\geq n + 1$, whence

$$\dim \Phi' \geq n.$$

§1.2. Retraction.

1.21. RETRACTION THEOREM. *Let Φ be a closed subset of a polyhedron Π and let K be a triangulation of Π. There exists a subcomplex K' of K and a continuous mapping C of the compactum Φ onto the polyhedron $\| K' \|$ with the following property: if $x \in \Phi$, there is a simplex $T \in K$ whose closure contains both points x and $C(x)$.*

Proof. Let $T_1 , \cdots , T_s$ be all the simplexes of K containing points of Φ enumerated in an order such that the dimension of T_{i+1} does not exceed the dimension of T_i.

Setting $\Phi_0 = \Phi$, let us denote by C_0 the identity mapping of Φ_0 and suppose that the set Φ_i and the mapping C_i have already been defined. If $T_{i+1} \subseteq \Phi_i$, set $\Phi_{i+1} = \Phi_i$ and denote by C_{i+1} the identity mapping of Φ_i.

If $T_{i+1} \not\subseteq \Phi_i$, take a point $o_{i+1} \in T_{i+1} \setminus \Phi_i$ and denote by C_{i+1} the *retraction of* Φ_i *onto* $\Phi_i \setminus T_{i+1}$, defined on $\Phi_i \setminus T_{i+1}$ as the identity and on $\Phi_i \cap T_{i+1}$ as the central projection of $\Phi_i \cap T_{i+1}$ from the point o_{i+1} into that part of the boundary $\overline{T}_{i+1} \setminus T_{i+1}$ of T_{i+1} which is contained in Φ_i. Let $\Phi_{i+1} = C_{i+1}(\Phi_i)$. Proceeding in this way step by step, we finally arrive at a compactum Φ_s, which is the body of some subcomplex K' of K, and a mapping

$$C = C_s C_{s-1} \cdots C_0,$$

which is easily seen to be the desired mapping of Φ onto the polyhedron $\| K' \| = \Phi_s$.

1.22. *Let* Φ *be a closed subset of a polyhedron* Π; *for every* $\epsilon > 0$ *there exists an* ϵ*-displacement* C *of the compactum* Φ *into the polyhedron* Π *which maps* Φ *onto a polyhedron* Π', *the body of some subcomplex* K' *of some triangulation* K *of* Φ.

To prove this, it suffices to take a triangulation K of the polyhedron Π of mesh $<\epsilon$, and to apply Theorem 1.21.

From 1.141 and 1.22 it follows that:

1.23. *Every compact subset of the Hilbert parallelotope can be mapped onto a polyhedron by means of an* ϵ*-displacement for every* $\epsilon > 0$.

From 1.142 and 1.23 it easily follows that:

1.24. *Every compactum can be* ϵ*-mapped onto a polyhedron for every* $\epsilon > 0$.

1.25. *In order that a compactum* Φ *in* R^n *contain interior points* (relative to R^n) *it is necessary and sufficient that*

$$\dim \Phi = n.$$

Indeed, since $\Phi \subset R^n$,

$$\dim \Phi \leq n;$$

if Φ contains interior points relative to R^n and consequently some n-dimensional simplex T^n, then $\dim \Phi \geq \dim T^n = n$ so that $\dim \Phi = n$.

Now suppose that the compactum $\Phi \subset R^n$ does not contain any interior points.

Since Φ is a closed bounded set in R^n, there exists a simplex $T^n \supset \Phi$.

Take a triangulation K of the closed simplex $\overline{T}^n$ of mesh less than a given ϵ. Since Φ does not contain any n-simplex, in particular, any n-simplex of the triangulation K, each of the n-simplexes $T^n{}_i \in K$ contains a point o_i not belonging to Φ. Let us now apply the retraction of 1.21, i.e., let us map the set $\Phi \cap T^n{}_i$ into $\overline{T}^n{}_i \setminus T^n{}_i$ by means of a central projection from the point o_i. Since this leaves all the points of Φ contained in $\overline{T}^n{}_i \setminus T^n{}_i$ fixed, the composite of the retractions defined in the various n-simplexes of

K is a continuous mapping of Φ into an $(n - 1)$-dimensional polyhedron which is the union of the boundaries $\overline{T}^n{}_i \setminus T^n{}_i$ of all the n-simplexes of K. Hence it follows that for every $\epsilon > 0$ there exists an ϵ-mapping of the compactum Φ into some $(n - 1)$-dimensional polyhedron. Hence, by virtue of Theorem 1.16,

$$\dim \Phi \leq n - 1.$$

This completes the proof of 1.25.

§1.3. The barycentric mapping of a space into the nerve of an open covering. Let

$$\omega = \{O_1, \cdots, O_s\}$$

be an open covering of a metric space X and let K_ω be a simplicial complex in some R^n isomorphic to the nerve of the covering ω. The elements of K_ω are simplexes (perhaps degenerate) of R^n.

The union $\| K_\omega \|$ of all the nondegenerate simplexes and the closed convex hulls of the degenerate simplexes of K_ω will be called the body of the complex K_ω. Since K_ω is an unrestricted complex, its body, in the sense just explained, can also be defined as the union of the closed convex hulls of all the degenerate and nondegenerate simplexes of K_ω.

Let us define in X the continuous functions μ_i, $i = 1, 2, \cdots, s$, by setting

$$\mu_i(x) = \rho(x, X \setminus O_i), \qquad x \in X,$$

and let us construct a continuous mapping C_ω of X into R^n as follows: let x be an arbitrary point of X; the vertices $a_1, \cdots, a_s$ of K_ω corresponding to the elements $O_1, \cdots, O_s$ of the covering ω are assigned the weights $\mu_1(x), \cdots, \mu_s(x)$, respectively; the centroid of these masses is then, by definition, the point $C_\omega(x)$. Since the functions $\mu_i(x)$ are continuous, C_ω is a continuous mapping of X into R^n. Let us prove that C_ω is *a mapping of X into the body of K_ω*. To this end, we note that the function $\mu_i(x)$ does not vanish only at points of the set O_i. Therefore, if $x \in X$ and x is contained in O_{i_j} $(0 \leq j \leq r)$, and only in these elements of ω, then $\mu_{i_j}(x) \neq 0$ $(0 \leq j \leq r)$ and all the remaining functions μ_i vanish at x. Since $x \in \bigcap_{j=0}^r O_{i_j}$, K_ω contains a simplex with vertices a_{i_j} $(0 \leq j \leq r)$ and $C_\omega(x)$ is obviously contained in the closed convex hull of the skeleton $\{a_{i_j}(0 \leq j \leq r)\}$. This proves the assertion.

DEFINITION 1.31. The continuous mapping C_ω of X into $\| K_\omega \|$ just constructed is called the *barycentric mapping* of X corresponding to the covering ω and the given choice of the vertices of the nerve K_ω of ω.

We shall now give several applications of barycentric mappings.

§1.4. Theorem on ϵ-mappings. In the notation of the preceding article,

let the order of the covering ω be equal to $r + 1$. Then if n is sufficiently large, namely if $n \geq 2r + 1$, choosing in R^n vertices $a_1, \cdots, a_s$ in general position yields a triangulation K_ω isomorphic to the nerve of the covering ω. Under these conditions, Theorems 1.41 and 1.42 hold:

1.41. *The inverse image of the simplex* $T^r = (a_{i_0} \cdots a_{i_r}) \in K_\omega$ *under the barycentric mapping* C_ω *of* X *into* $\| K_\omega \|$ *is*

$$C_\omega^{-1}(a_{i_0} \cdots a_{i_r}) = \bigcap_{j=0}^{r} O_{i_j} \setminus \bigcup' O_i ,$$

where $\bigcup' O_i$ is the union of all O_i for which $i \neq i_0, \cdots, i_r$.

Indeed, since K_ω is a triangulation, the various simplexes $T_i \in K_\omega$ are mutually disjoint; therefore, if a point $\nu = C_\omega(x)$ is contained in a simplex $(a_{i_0} \cdots a_{i_r})$, it is not contained in any other simplex of K_ω; in other words, $\mu_{i_j}(x) > 0$ $(0 \leq j \leq r)$ and $\mu_i(x) = 0$ $(i \neq i_0, \cdots, i_r)$. But this means that

$$x \in \bigcap_{j=0}^{r} O_{i_j} \setminus \bigcup' O_i .$$

1.42. If ω is an ϵ-covering, $\epsilon > 0$, then the barycentric mapping C_ω is an ϵ-mapping of the space X into the polyhedron $\| K_\omega \|$, whose dimension is one less than the order of the covering ω.

Hence, by Theorem 1.16, it follows that:

1.43. Every r-dimensional compactum can be ϵ-mapped into some r-dimensional polyhedron for every $\epsilon > 0$ and cannot be ϵ-mapped into any polyhedron of smaller dimension for a sufficiently small ϵ.

This result may be strengthened as follows:

Let $\omega = \{O_1, \cdots, O_s\}$ be an open ϵ-covering of order $r + 1$ of a compactum X of dimension r. Let the triangulation K_ω situated in some R^n be isomorphic to the nerve of the covering ω. Let us consider the barycentric mapping C_ω of the compactum X into $\| K_\omega \|$; by Theorem 1.42, C_ω is an ϵ-mapping into $\| K_\omega \|$. Let us now subject the set $C_\omega(X) \subseteq \| K_\omega \|$ to a retraction C' (1.21). The result is a closed subcomplex K'_ω of K_ω and a continuous mapping $C'C_\omega$ of X onto $\| K'_\omega \|$. Recalling the definition of C', it is easy to see that $C'C_\omega$ can map onto any point y of a simplex $T' = (a_{i_0} \cdots a_{i_r})$ of K'_ω only a point $x \in X$ which is mapped by C_ω into some simplex T having T' as a face and consequently having the points $a_{i_0}, \cdots, a_{i_r}$ among its vertices. Hence the inverse image of the point $y \in T'$ under the mapping $C'C_\omega$ is a subset of $\bigcap_{j=0}^{r} O_{i_j}$, so that $(C'C_\omega)^{-1}y$ has diameter $<\epsilon$.

Hence

Every r-dimensional compactum can be ϵ-mapped onto some r-dimensional polyhedron for every $\epsilon > 0$ and cannot be ϵ-mapped into any polyhedron of dimension $<r$ for sufficiently small ϵ.

In other words;

1.44. *The dimension of a compactum Φ may be defined as the least r with the property that for every $\epsilon > 0$ there is an ϵ-mapping of Φ onto a polyhedron of dimension number r. If there is no number r with this property, the dimension of Φ is infinite.*

§1.5. Barycentric approximations of a given continuous mapping of a compactum Φ into R^n. Theorem on ϵ-displacements. Let C be a continuous mapping of a compactum Φ into R^n and let $\epsilon > 0$. Let $\sigma > 0$ be so small that $\rho(x, x') < \sigma$ (in Φ) implies that $\rho(Cx, Cx') < \epsilon/4$ (in R^n); for the rest, σ is perfectly arbitrary. Let

$$\omega = \{O_1, \cdots, O_s\}$$

be an open σ-covering of Φ. In each set O_i choose a point o_i; choose mutually distinct points $a_i \in R^n$ to satisfy the condition

$$\rho(Co_i, a_i) < \epsilon/4 \tag{1.50}$$

and make them the vertices of the nerve K_ω of the covering ω. By the simplexes of the complex K_ω we shall again understand the simplexes (degenerate or not) defined by the corresponding skeletons of K_ω. The body $\| K_\omega \|$ of the complex K_ω and the barycentric mapping C_ω of Φ into $\| K_\omega \|$ is defined as in 1.3. Let us call this mapping the *barycentric ϵ-approximation of the mapping C* (corresponding to the covering ω and the choice of the vertices a_i). This name is justified by the fact that for every $x \in \Phi$

$$\rho(Cx, C_\omega x) < \epsilon. \tag{1.51}$$

Indeed, let x be an arbitrary point of Φ. Let $O_{i_0}, \cdots, O_{i_r}$ be all the elements of ω containing x. Then the distance of the point x from each of the points $o_{i_0}, \cdots, o_{i_r}$ is less than σ, so that the distance of the point Cx from each of the points $Co_{i_0}, \cdots, Co_{i_r}$ is less than $\epsilon/4$. In virtue of (1.50), $\rho(Cx, a_i) < \epsilon/2, i = i_0, \cdots, i_r$. In other words, all the points $a_{i_0}, \cdots, a_{i_r}$ are contained in the interior of a sphere of radius $\epsilon/2$ with center Cx. The interior of this sphere contains the closed convex hull of $\{a_{i_0}, \cdots, a_{i_r}\}$. Hence the open sphere also contains $C_\omega x$. This proves (1.51).

Let $\dim \Phi = r$. Then the covering ω can be taken of order $r + 1$ so that K_ω is an r-complex and $\| K_\omega \|$ is the union of a finite number of closed simplexes of dimension $\leq r$ (in virtue of IV, 3.1, Remark 2, $\| K_\omega \|$ is an r-dimensional polyhedron, but we do not wish to use this fact here).

Hence

1.51. *For every continuous mapping C of an r-dimensional compactum Φ into R^n there exists a continuous mapping C_ω of Φ into a subset $\| K_\omega \|$ of the given R^n, differing from C by as little as desired. $\| K_\omega \|$ is the union of a finite number of closed simplexes of dimension $\leq r$. The mapping C_ω*

can be taken as the barycentric approximation C_ω of the mapping C constructed by means of an arbitrary sufficiently fine open covering ω of Φ having order $r + 1$.

REMARK. If C maps Φ into the sphere $\bar{E}^n$, we may also assume that $\| K_\omega \| \subseteq \bar{E}^n$. Indeed, if $\rho(C, C_\omega)$ is less than some given ϵ, K_ω is contained in a sphere $\bar{E}^n{}_0$ of radius $\rho + \epsilon$ concentric with $\bar{E}^n$, where ρ is the radius of $\bar{E}^n$. Let C' be the similitude which takes $\bar{E}^n{}_0$ onto $\bar{E}^n$; C' maps each of the simplexes of K_ω into a simplex contained in $\bar{E}^n$ and the union of these simplexes is $\| K'_\omega \| = C'(\| K_\omega \|) \subset \bar{E}^n$. The mapping $C'C_\omega$ differs from C by less than $\epsilon + \epsilon = 2\epsilon$ and transforms Φ into $\| K'_\omega \| \subset \bar{E}^n$. We note in this connection that if C_ω is a σ-mapping (for some $\sigma > 0$), then, since C' is (1–1), $C'C_\omega$ is also a σ-mapping.

As before, let $\dim \Phi = r$ and, in addition, let $n \geq 2r + 1$. The positive number ϵ is arbitrary, σ sufficiently small: namely, $\rho(x, x') < \sigma$ in Φ implies $\rho(Cx, Cx') < \epsilon/4$ in R^n. Let a σ-covering ω of Φ have order $r + 1$. Choosing the vertices $a_1, \cdots, a_s$ in R^n in general position, we obtain an r-dimensional triangulation K_ω. The mapping C_ω satisfies both (1.51) and the hypothesis of Theorem 1.41 (stated at the beginning of 1.4) and is therefore a σ-mapping.

Hence, applying the remark just made, we have

1.52. *Let C be an arbitrary continuous mapping of an r-dimensional compactum Φ into the sphere $\bar{E}^n$, where $n \geq 2r + 1$. For every positive ϵ and σ, there exists a σ-mapping C_ω of Φ into some r-dimensional polyhedron $\| K \| \subseteq \bar{E}^n$ such that $\rho(Cx, C_\omega x) < \epsilon$ for all $x \in \Phi$.*

The following proposition is essential for the sequel. It follows at once from 1.52.

1.520. *Let $n \geq 2r + 1$. For every $\sigma > 0$ the set of all σ-mappings of an r-dimensional compactum Φ into the closed solid n-sphere $\bar{E}^n$ is dense in the space $\mathfrak{C}(\Phi, \bar{E}^n)$ of all continuous mappings of Φ into $\bar{E}^n$.*

Let us note yet another corollary of Theorem 1.52. Let C be the identity mapping; then the mapping C_ω is an ϵ-displacement and we obtain the following result:

1.521. *For every r-dimensional compactum $\Phi \subseteq R^n$ and for every $\epsilon > 0$ there is an ϵ-displacement of Φ onto some r-dimensional polyhedron $\| K_\omega \|$ situated in the space R^{2n+1} containing the given space R^n.*

REMARK 1. Let R^m be an arbitrary Euclidean space containing the r-dimensional compactum Φ. If $m \geq 2r + 1$, set $n = m$; if $m \leq 2r + 1$, set $n = 2r + 1$. Let us choose the vertices $a_i \in R^n$ of the complex K_ω in an arbitrarily given neighborhood of $R^m \subseteq R^n$.

Let us apply Theorem 1.521 to produce an ϵ-displacement of the compactum Φ into $\| K_\omega \|$. Returning the vertices of K_ω to the space R^m by a small displacement C_1, we obtain a mapping C_1 of the polyhedron

$\| K_\omega \|$ onto the set $C_1(\| K_\omega \|)$ which is the union of a finite number of closed simplexes [of the form $C_1(\overline{T})$, where $T \in K_\omega$].

Since these simplexes have dimension $\leq r$ and at least one is an r-simplex, $C_1(\| K_\omega \|)$ is an r-dimensional polyhedron (IV, 3.1, Remark 2). Hence

1.522. *For every r-dimensional compactum* $\Phi \subset R^n$ *and for every* $\epsilon > 0$ *there exists an* ϵ*-displacement of* Φ *onto an r-dimensional polyhedron* $\subset R^n$.

Finally, let Φ be an r-dimensional compactum in the Hilbert parallelotope. Let $\epsilon > 0$ and let

$$\omega = \{O_1, \cdots, O_s\}$$

be an open ϵ-covering of order $r + 1$ of Φ.

Let C be an ϵ-displacement of Φ into some R^n, $n \geq 2r + 1$. A barycentric approximation C_ω of C can be chosen so that it will be a mapping of Φ into a polyhedron $\| K_\omega \| \subset R^n$ whose triangulation K_ω is isomorphic to the nerve of the covering ω and has mesh $<2\epsilon$.

The mapping C_ω differs from the ϵ-displacement C by less than ϵ and is therefore a 2ϵ-displacement.

Finally, subjecting $C_\omega(\Phi) \subseteq \| K_\omega \|$ to the retraction C', which under our conditions is a 2ϵ-displacement, we obtain a 4ϵ-displacement $C'C_\omega$ of Φ onto a polyhedron $\| K'_\omega \|$, $K'_\omega \subseteq K_\omega$, whose dimension is $\leq r$. For a sufficiently small ϵ the dimension of the polyhedron $\| K'_\omega \|$, by 1.16, cannot be less than r and hence is equal to r.

Therefore

1.530. For every r-dimensional compactum Φ contained in the Hilbert parallelotope and for every $\epsilon > 0$ there is an ϵ-displacement of Φ onto some r-dimensional polyhedron.

From what has been proved it follows that:

1.53. *The dimension of a compactum* Φ *contained in the Hilbert parallelotope can be defined as the least number r satisfying the following condition: for every* $\epsilon > 0$ *there exists an* ϵ*-displacement of the compactum* Φ *into the Hilbert parallelotope mapping* Φ *onto some r-dimensional polyhedron.*

REMARK 2. From the proofs of the theorems of this article it follows that it is always possible to take for the polyhedron onto which Φ is mapped by means of an ϵ-displacement the body of some subcomplex of the nerve of every sufficiently fine open (or, by I, 8.331, closed) covering of Φ; the nerve itself must be constructed in a sufficiently small neighborhood of the given covering ω (IV, 2.1, Remark 4).

§1.6. Theorem on imbedding r-dimensional compacta in R^{2r+1}.

We shall prove the following fundamental theorem:

THEOREM 1.6. *Every r-dimensional compactum is homeomorphic to a subset of* R^{2r+1}.

The proof is based on Theorem 1.520 and on the following lemma:

LEMMA 1.61. *Let X and Y be arbitrary compacta and let $\epsilon > 0$; let $\mathfrak{C}(X, Y)$ be the space of continuous mappings of X into Y. The set of all ϵ-mappings of X into Y is open in $\mathfrak{C}(X, Y)$.*

It suffices to prove that every mapping $C' \in \mathfrak{C}(X, Y)$ which is sufficiently near [in terms of the metric in $\mathfrak{C}(X, Y)$] an ϵ-mapping C, is itself an ϵ-mapping. Let us define for the number ϵ and the given ϵ-mapping C a number η in accordance with Lemma 1.15 and assume that C' satisfies the inequality $\rho(C, C') < \eta/2$. Now let x_1 and x_2 be two points of X, mapped by C' into the same point $y \in Y$. Then

$$\begin{aligned}\rho(Cx_1, Cx_2) &\leq \rho(Cx_1, C'x_1) + \rho(C'x_1, C'x_2) + \rho(C'x_2, Cx_2) \\ &< \eta/2 + 0 + \eta/2 = \eta,\end{aligned}$$

whence, by the definition of η, $\rho(x_1, x_2) < \epsilon$, i.e., C' is an ϵ-mapping. This proves Lemma 1.61.

Let us now pass to the proof of Theorem 1.6.

For every natural number m denote by Γ_m the set of all $(1/m)$-mappings of an r-dimensional compactum Φ into the $(2r + 1)$-dimensional closed solid sphere $\bar{E}^n$. By Lemma 1.61, Γ_m is an open subset of $\mathfrak{C}(\Phi, \bar{E}^n)$; by Theorem 1.520, Γ_m is dense in $\mathfrak{C}(\Phi, \bar{E}^n)$. By I, 7.31, $\mathfrak{C}(\Phi, \bar{E}^n)$ is a complete metric space. Consequently in virtue of I, 7.14, the intersection of all the Γ_m is a nonvacuous set $\mathfrak{C}$ [this set is even dense in $\mathfrak{C}(\Phi, \bar{E}^n)$]. The elements of the set $\mathfrak{C}$ are continuous mappings of Φ into $\bar{E}^n$, each of which is an ϵ-mapping for arbitrary $\epsilon > 0$. Hence it follows that all the elements of the set $\mathfrak{C}$ are (1–1) mappings of Φ into $\bar{E}^n$, and since Φ and $\bar{E}^n$ are compacta, all these mappings are topological, q.e.d.

REMARK 1. The proof of the following theorem is implicit in the above arguments:

THEOREM 1.62. *For every continuous mapping C of a compactum Φ of dimension r into the Euclidean space R^n, where $n \geq 2r + 1$, and for every $\epsilon > 0$ there exists a topological mapping C_0 such that*

$$\rho(Cx, C_0x) < \epsilon$$

for all $x \in \Phi$.

REMARK 2. From Theorem 1.6. it follows that among the topological spaces the finite dimensional compacta are characterized as those which are homeomorphic to closed bounded subsets of Euclidean spaces.

§2. Theorem on essential mappings

§2.1. Definition and statement of the theorem. Let C be a continuous mapping of a compactum Φ onto a closed n-cell $\bar{E}^n$ with boundary $S^{n-1} =$

$\bar{E}^n \setminus E^n$ [V, Theorem 3.16 (in the sequel $\bar{E}^n$ will be either a closed n-simplex or a solid n-sphere)]. Let Φ_0 denote the inverse image of S^{n-1} under the mapping C, i.e. the set of all points of Φ mapped by C into S^{n-1}. The mapping C is said to be *essential* if every continuous mapping C_1 of Φ into $\bar{E}^n$ coinciding with C on Φ_0 is a mapping *onto* $\bar{E}^n$.

This definition implies that if C is not an essential mapping, there exists a continuous mapping C_1 of Φ into $\bar{E}^n$ coinciding with C on Φ_0 such that some interior point o of $\bar{E}^n$ is not contained in $C_1(\Phi)$. [Indeed, if the entire interior E^n of $\bar{E}^n$ were contained in $C_1(\Phi)$, then, since $C_1(\Phi)$ is closed, we would have $C_1(\Phi) = \bar{E}^n$, i.e., C_1 would be mapped onto $\bar{E}^n$.]

This definition of essential mappings is very simple but not very intuitive: the intuitive meaning of essential mappings is clarified in the very best way by the following remark:

2.11. Let us call a deformation C_θ of a mapping $C = C_0$, $0 \le \theta \le 1$, for which all the mappings C_θ coincide with the mapping C on Φ_0, an *admissible deformation. The mapping C_0 is inessential if, and only if, there is an admissible deformation of C_0 into a mapping C_1 such that*

$$C_1(\Phi) \subseteq S^{n-1}.$$

Proof of 2.11. Without affecting the generality of the proof, we may assume that $\bar{E}^n$ is a solid sphere bounded by S^{n-1}.

If an admissible deformation C_θ exists, the mapping C_0 is obviously inessential in the sense of the definition given at the beginning of this article.

Now let C_0 be inessential in the first sense. Then there exists a mapping, which we shall denote by $C_{\frac{1}{2}}$, coinciding with C_0 on Φ_0 and such that some point $o \in \bar{E}^n$ is not contained in the set $C_{\frac{1}{2}}(\Phi)$. Let us note first that there is an admissible deformation of C_0 into $C_{\frac{1}{2}}$; it suffices, for arbitrary $x \in \Phi$ and arbitrary θ, $0 \le \theta \le \frac{1}{2}$, to denote by $C_\theta x$ the point which divides the directed segment $(C_0 x, C_{\frac{1}{2}} x)$ in the ratio θ: $(\frac{1}{2} - \theta)$.

Let us now denote by C_θ, $\frac{1}{2} \le \theta \le 1$, the deformation of the mapping $C_{\frac{1}{2}}$ defined by a central projection of the set $C_{\frac{1}{2}}(\Phi)$ from the point o into S^{n-1}: for each $x \in \Phi$ and θ, $\frac{1}{2} \le \theta \le 1$, construct the ray $oC_{\frac{1}{2}}(x)$, denote by $C_1(x)$ the point of intersection of this ray with the sphere S^{n-1}, and by $C_\theta(x)$ the point which divides the segment $C_{\frac{1}{2}}(x)C_1(x)$ in the ratio $(\theta - \frac{1}{2})$: $(1 - \theta)$. The resulting deformation C_θ, $0 \le \theta \le 1$, is the desired admissible deformation.

We shall prove the following proposition in this section. This proposition, as also the theorem on ϵ-mappings (on ϵ-displacements) proved in the preceding section, is one of the fundamental theorems of dimension theory.

THEOREM 2.1. *An r-dimensional compactum can be mapped essentially*

onto a closed r-simplex; every continuous mapping of an r-dimensional compactum onto a closed n-cell is inessential for $n > r$.

REMARK 1. In particular, every continuous mapping of a segment onto a square (Peano curve) is inessential.

REMARK 2. Theorem 2.1 implies that the dimension of a compactum Φ may be defined as the maximum r for which it is possible to map Φ essentially onto a closed r-simplex or, equivalently, onto a solid r-sphere.

§2.2. **Proof of Theorem 2.1.** We shall prove the second assertion of Theorem 2.1 first: *for* $n > r$ *every continuous mapping of an r-dimensional compactum* Φ *onto an n-cell* $\bar{E}^n$ *is inessential.* Without affecting the generality of the result it may obviously be assumed that $\bar{E}^n$ is a closed solid n-sphere.

Let C be a mapping of Φ onto $\bar{E}^n$; by Theorem 1.51 (and the remark to this theorem) there is a mapping C_ω, which differs from C by as little as desired, of Φ onto a set $\| K_\omega \| \subset \bar{E}^n$ which is the union of a finite number of closed simplexes of dimension $\leq r$; since $r < n$, the set $\| K_\omega \|$ is nowhere dense in $\bar{E}^n$. Hence, if we prove the following proposition, it will follow that C is inessential:

2.21. *Let C be an essential mapping of a compactum* Φ *into a solid n-sphere* $\bar{E}^n$; *there exists an* $\epsilon > 0$ *such that for every continuous mapping* C_1 *of* Φ *into* $\bar{E}^n$ *differing from C by less than* ϵ [i.e., such that $\rho(Cx, C_1x) < \epsilon$ for all $x \in \Phi$] *there is a solid n-sphere* $\bar{L}^n_1$ *concentric with* $\bar{E}^n$ *contained in* $C_1(\Phi)$.

We shall prove the stronger proposition:

2.22. *Let C be an essential mapping of a compactum* Φ *onto a solid n-sphere* $\bar{E}^n$ *of radius* 1 *with center o. Let* $\bar{E}^n{}_{\frac{1}{2}}$ *be the solid sphere of radius* $\frac{1}{2}$ *concentric with the sphere* $\bar{E}^n$.

Let S^{n-1} *be the bounding sphere of* $\bar{E}^n$ *and* $\Phi_0 = C^{-1}(S^{n-1})$.

If $\epsilon < \frac{1}{2}$, *then for every continuous mapping* C_1 *of* Φ *into* $\bar{E}^n$ *satisfying the condition*

$$\rho(Cx, C_1x) < \epsilon \qquad \textit{for all } x \in \Phi_0,$$

we have $\bar{E}^n{}_{\frac{1}{2}} \subseteq C_1(\Phi)$.

Proof of 2.22. Let $\epsilon < \frac{1}{2}$; let $\bar{E}^n{}_r$ be the solid sphere of radius r with center o. The mapping C_1 by hypothesis differs from $C = C_0$ by less than ϵ at all $x \in \Phi_0$. Let x be an arbitrary point of Φ and $0 \leq \theta \leq 1$; denote by $C_\theta x$ the point which divides the segment $C_0(x)C_1(x)$ in the ratio $\theta:(1 - \theta)$. Let us now construct the solid sphere $\bar{E}^n{}_r$ of radius $r < 1$ differing from 1 by so little that for every $x \in \Phi$ for which

$$\rho(o, Cx) \geq r$$

the segment $C(x)C_1(x)$ is in the exterior of the sphere $\bar{E}^n{}_{\frac{1}{2}}$. Denote the boundary of the sphere $\bar{E}^n{}_r$ by $S_r{}^{n-1}$. Let us define mappings C'_θ, $0 \leq \theta \leq 1$, as follows:

1. If $\rho(o, Cx) \geq r$, extend the ray $oC(x)$ and consider the segment Δ of this ray included between S^{n-1} and S_r^{n-1} directed *to the center* o of the spheres. Then $C'_\theta x$ is the point of the segment $C(x)C_\theta(x)$ dividing this segment in the same ratio in which the point Cx divides the directed segment Δ. In particular, if $Cx \in S^{n-1}$, then $C'_\theta x = Cx$ for all θ.

On the other hand, if $Cx \in S_r^{n-1}$, $C'_\theta x = C_\theta x$.

2. If Cx is inside the solid sphere E^n_r, then $C'_\theta x = C_\theta x$. Since C'_0 obviously coincides with $C_0 = C$, the family of mappings C'_θ is a deformation of the mapping C such that $C'_\theta x = Cx$ for all $x \in \Phi_0$ and for arbitrary θ. In other words, C'_θ is an admissible deformation of the essential mapping C. Therefore all of the mappings C'_θ are mappings of Φ onto $\bar{E}^n$. A fortiori, the entire sphere $\bar{E}^n{}_{\frac{1}{2}}$ is contained in $C'_\theta(\Phi)$ for arbitrary θ. The mappings C'_θ can take only those points of Φ into $\bar{E}^n{}_{\frac{1}{2}}$ which C maps into $\bar{E}^n{}_r$; but the mappings C_θ and C'_θ coincide in these points. Hence $C_\theta(\Phi) \supseteq \bar{E}^n{}_{\frac{1}{2}}$. In particular, $C_1(\Phi) \supseteq \bar{E}^n{}_{\frac{1}{2}}$. This proves proposition 2.22 and consequently the second assertion of Theorem 2.1.

Let us now go on to the first assertion of Theorem 2.1.

Every n-dimensional compactum Φ can be mapped essentially onto a closed n-simplex.

Proof. Let us take $\epsilon > 0$ so small that Φ cannot be 2ϵ-mapped into any polyhedron of dimension $<n$.

Let C_0 be an ϵ-mapping of Φ onto an n-dimensional polyhedron Π. Let us define the number η in accordance with Lemma 1.15 and consider a triangulation K^n of the polyhedron Π of mesh $<\eta/2$. Let $T^n{}_1, \cdots, T^n{}_s$ be all the n-simplexes of the triangulation. Set

$$\Phi_i = C^{-1}{}_0(\overline{T}^n{}_i), \qquad \Phi_{0i} = C^{-1}{}_0(\overline{T}^n{}_i \setminus T^n{}_i)$$

and define the mapping C_{0i} of Φ_i onto $\overline{T}^n{}_i$ as the mapping defined by C_0 on Φ_i.

We shall prove that the mapping C_{0i} is essential for at least one i. Suppose the contrary; then for every i there exists an admissible deformation $C_{\theta i}$, $0 \leq \theta \leq 1$, taking the mapping C_{0i} into a mapping C_{1i} of Φ_i into $\overline{T}^n{}_i \setminus T^n{}_i$. $C_{\theta i} = C_0$ on all the sets Φ_{0i} and hence at all the points contained in more than one Φ_i, for arbitrary θ and for arbitrary i; hence setting

$$C_\theta(x) = C_0(x), \quad \text{if} \quad x \in \textstyle\bigcup_i \Phi_{0i}$$

and

$$C_\theta(x) = C_{\theta i}(x), \quad \text{if} \quad x \in \Phi_i$$

for arbitrary θ, $0 \leq \theta \leq 1$, we obtain a deformation C_θ of the mapping $C = C_0$ of Φ, where C_1 maps Φ into the $(n-1)$-dimensional polyhedron $\| K^{n-1} \| = \| K^n \| \setminus (T^n{}_1 \cup \cdots \cup T^n{}_s)$. The deformation C_θ has the following property: if $T \in K^n$ is the carrier of the point $C_0(x)$, then for arbitrary

θ and in particular for $\theta = 1$, the point $C_\theta(x)$ is contained in $\overline{T}$. Hence it follows that C_1 can map into a given point $y \in T \in K^{n-1}$ only those points of Φ which are mapped by C_0 into points of simplexes having the simplex T as a face; in other words, denoting by Γ the body of the star $O_{K^n}T$, we have

$$C^{-1}{}_1(x) = C^{-1}{}_0(\Gamma).$$

Since the diameter of Γ is less than η, the set $C^{-1}{}_0(\Gamma)$, and therefore $C^{-1}{}_1(\Gamma)$, has diameter $<\epsilon$, i.e., C_1 is an ϵ-mapping of Φ into the $(n - 1)$-dimensional polyhedron $\| K^{n-1} \|$. This contradicts the choice of the number ϵ.

Hence $C_{0i} = C_0$ is essential on at least one of the sets Φ_i, say on Φ_1. Now assign to each vertex of the simplex $T^n{}_1$ this vertex itself; to each vertex of the complex K^n not belonging to the simplex $T^n{}_1$ assign a vertex of the simplex $T^n{}_1$. This yields a simplicial mapping C' of the polyhedron Π onto the closed simplex $\overline{T}^n{}_1$, such that C' is the identity mapping on $\overline{T}^n{}_1 \subseteq \Pi$. Hence it follows easily that the mapping $C'C$ is an essential mapping of Φ onto the closed n-simplex $T^n{}_1$, q.e.d.

§3. The sum theorem. Inductive dimension

§3.1. Statement and proof of the sum theorem.

THEOREM 3.1. *Let there be given in a compactum Φ a finite number of closed sets; if the dimension of each of these sets does not exceed n, then the dimension of their union does not exceed n.*

It suffices to prove Theorem 3.1 for the case of two sets. Hence let A_1 and A_2 be two closed subsets of Φ of dimension $\leq n$. We shall prove that $A = A_1 \cup A_2$ has dimension $\leq n$.

Hence it is necessary to prove that for arbitrary $\epsilon > 0$ there exists a closed ϵ-covering of A of order $\leq n + 1$.

Let us take an ϵ-covering

$$\alpha_0 = \{A^0{}_1, \cdots, A^0{}_{s(0)}\}$$

of order $\leq n + 1$ of the set $A_0 = A_1 \cap A_2$ and denote by 2σ a Lebesgue number of the covering α_0. Let

$$\alpha_\lambda = \{A^\lambda{}_1, \cdots, A^\lambda{}_{s(\lambda)}\}$$

be a closed σ-covering of order $\leq n + 1$ of A_λ, $\lambda = 1, 2$. Let

$$A^\lambda{}_1, \cdots, A^\lambda{}_{u(\lambda)}$$

be all the elements of the covering α_λ which do not meet the set A_0. Further denote by $A^\lambda{}_{1,1}, \cdots, A^\lambda{}_{1,m(\lambda,1)}$ all the elements of α_λ which intersect $A^0{}_1$ and by $A^\lambda{}_{i,1}, \cdots, A^\lambda{}_{i,m(\lambda,i)}$ all the elements of α_λ which intersect

A^0_i $(i \geq 2)$ but do not intersect a single one of the sets $A^0_1, \cdots, A^0_{i-1}$. Let us set

$$B^\lambda_i = A^\lambda_i, \qquad i = 1, 2, \cdots, u(\lambda),$$

and

$$B^0_i = A^1_{i,1} \cup \cdots \cup A^1_{i,m(1,i)} \cup A^2_{i,1} \cup \cdots \cup A^2_{i,m(2,i)},$$

$$i = 1, 2, \cdots, s(0);$$

some of the B^0_i may turn out to be empty.

The sets B^0_i, B^1_i, B^2_i obviously form a closed ϵ-covering β of the set A. It remains to be proved that the order of the covering β does not exceed $n + 1$. To this end we note first that no B^1_i intersects any B^2_k since a point contained in B^1_i and B^2_k would be contained in A_0. This is impossible since neither B^1_i nor B^2_k intersects A_0 by definition.

Further, α_0, α_1, α_2 are systems of sets each of which has order $\leq n + 1$; besides 2σ is a Lebesgue number of the system α_0 and $B^0_i \subset S(A^0_i, \sigma)$. Hence it follows that there cannot exist more than $n + 1$ sets of each of the types B^0_i, B^1_i, B^2_i with nonempty intersection (see I, 8.331). Therefore, if the order of the system β exceeds $n + 1$, there exist $n + 2$ sets in β of the form

$$(3.11) \qquad B^0_{i_k}, B^1_{i_l}, \qquad k = 1, \cdots, \mu; \qquad l = \mu + 1, \cdots, n + 2,$$

or of the form

$$(3.12) \qquad B^0_{i_k}, B^2_{i_l}, \qquad k = 1, \cdots, \mu; \qquad l = \mu + 1, \cdots, n + 2,$$

with nonempty intersection. We shall prove that this is impossible. It suffices to consider one of the two cases, say (3.11). Hence suppose there exists a point p contained in all the sets (3.11). Since $p \in B^0_{i_k}$, p is contained in some set $A^\lambda_{i_k,\nu}$ intersecting $A^0_{i_k}$; in this connection, if $\lambda = 2$, $p \in A^2_{i_k,\nu}$ so that $p \in A_2$ and $p \in B^1_{i_{\mu+1}}$, i.e., $B^1_{i_{\mu+1}} \cap A_2 \neq 0$. Hence $B^1_{i_{\mu+1}} \cap A_0 \neq 0$, which is impossible. Hence

$$p \in \bigcap_{k=1}^{\mu} A^1_{i_k,\nu_k} \cap \bigcap_{l=\mu+1}^{n+2} B^1_{i_l},$$

i.e.,

$$p \in \bigcap_{k=1}^{\mu} A^1_{i_k,\nu_k} \cap \bigcap_{l=\mu+1}^{n+2} A^1_{i_l}.$$

Here all the sets $A^1_{i_k,\nu_k}$ are distinct, since the set $A^1_{i_k,\nu_k}$ meets $A^0_{i_k}$ and does not meet any of the sets A^0_j, $j < i_k$. Since all the $A^1_{i_{\mu+r}}$ are also distinct and different from all $A^1_{i_k,\nu_k}$, the point p is contained in $n + 2$ elements of the covering α_1, which is impossible.

This proves Theorem 3.1.

§3.2. Inductive dimension.

THEOREM 3.21. *If every point of a compactum* Φ *has a neighborhood of arbitrarily small diameter, whose boundary* (the boundary of an open set Γ in a topological space R is the closed set $\bar{\Gamma} \setminus \Gamma$) *has dimension* $\leq n - 1$, *then* $\dim \Phi \leq n$.

Proof. Given that each point $p \in \Phi$ has for a given arbitrarily small $\epsilon > 0$ a neighborhood $O(p)$ satisfying the conditions

1. $\delta[\bar{O}(p)] < \epsilon$,
2. $\dim [\bar{O}(p) \setminus O(p)] \leq n - 1$,

it is required to construct a closed ϵ-covering of Φ of order $\leq n + 1$.

Let us construct for every $p \in \Phi$ a neighborhood $O(p)$ satisfying the conditions 1 and 2 and choose from this collection a finite number of neighborhoods

$$O_1 = O(p_1), \cdots, O_s = O(p_s)$$

covering the whole space Φ.

By Theorem 3.1 the set

$$A' = (\bar{O}_1 \setminus O_1) \cup \cdots \cup (\bar{O}_s \setminus O_s)$$

has dimension $\leq n - 1$, so that there exists a closed ϵ-covering

$$\alpha' = \{A'_1, \cdots, A'_u\}$$

of order $\leq n$ of the set A'.

Let 2σ be a Lebesgue number of the system α' (in Φ). Denote by O'_i a σ-neighborhood of the set A'_i in Φ.

The union of all the O'_i is a neighborhood O' of the set A'. Let us now put

$$A_1 = \bar{O}_1 \setminus O',$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_i = \bar{O}_i \setminus (O_1 \cup \cdots \cup O_{i-1} \cup O'), \qquad i = 1, 2, \cdots, s,$$

and consider the system of sets

$$\alpha = \{A_1, \cdots, A_s, \bar{O}'_1, \cdots, \bar{O}'_u\}.$$

Since

$$A_1 \cup \cdots \cup A_s \cup O' = \Phi,$$

α is a (closed) covering of the space Φ, and this covering is obviously an ϵ-covering. Furthermore, since

$$\bar{O}_i = O_i \cup (\bar{O}_i \setminus O_i) \subseteq O_i \cup A' \subseteq O_i \cup O',$$

it follows that for $i < k$

$$A_i \cap A_k \subseteq \bar{O}_i \cap [\bar{O}_k \setminus (O_i \cup O')] \subseteq \bar{O}_i \cap (\bar{O}_k \setminus \bar{O}_i) = 0.$$

Therefore the order of the covering α is at most 1 greater than the order of the system $\bar{O}'_1, \cdots, \bar{O}'_u$ which in turn does not exceed n. Hence the order of the covering α is less than or equal to $n + 1$, q.e.d.

Theorem 3.21 gives a reason for defining for a topological space R a certain new topological invariant called the *inductive dimension* and denoted by ind R.

By definition, the only topological space whose inductive dimension is equal to -1 is the space which does not contain a single point (the empty set).

Let us suppose that we have already defined every space whose inductive dimension $\leq n - 1$, where n is a nonnegative integer. Then we say that *a topological space R has inductive dimension $\leq n$ if for every neighborhood Ox of an arbitrary point $x \in R$ there is a neighborhood $O_1x \subseteq Ox$ of x whose boundary has inductive dimension $\leq n - 1$.*

If a space R of inductive dimension $\leq n$ is not at the same time a space of inductive dimension $\leq n - 1$, we say that the inductive dimension of R is equal to n.

REMARK 1. For a metric space we can obviously insert the following change in this definition:

Assuming that the spaces having inductive dimension $\leq n - 1$ have already been defined, we say that a metric space R has inductive dimension $\leq n$ if for every point $x \in R$ and arbitrary $\epsilon > 0$ there is a neighborhood Ox of diameter $< \epsilon$ whose boundary has inductive dimension $\leq n - 1$.

REMARK 2. We shall say that a space has inductive dimension $\leq n$ at a given point x

$$\text{ind}_x R \leq n$$

if every neighborhood Ox of x contains a neighborhood O_1x of x whose boundary has inductive dimension $\leq n - 1$.

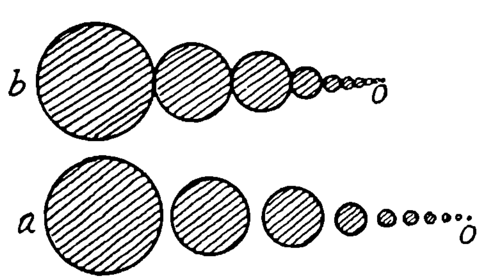


FIG. 98

If a space R has inductive dimension $\leq n$ at a point x, but not $\leq n - 1$, we shall say that $\text{ind}_x R = n$.

EXAMPLE 1. If R is the Cantor perfect set or the set of all irrational numbers, then $\text{ind}_x R = 0$ at each point x of R.

2. A compactum consisting of a point o and mutually disjoint circles converging to o (Fig. 98a) has inductive dimension 0 at the point o.

But if each circle is tangent externally to its successor (Fig. 98b) and R again consists of all these circles and of the point o to which they converge, then $\text{ind}_o R = 1$.

One of the basic propositions of dimension theory is the following:

THEOREM 3.22 (Urysohn). *If Φ is a compactum,*

$$\dim \Phi = \text{ind}\, \Phi. \tag{3.22}$$

In this section we shall prove only the inequality

$$\dim \Phi \leq \text{ind}\, \Phi. \tag{3.221}$$

This inequality is easily extended to the case of Φ an arbitrary bicompactum (see Aleksandrov [e]); it is not yet known whether (3.22) holds for arbitrary bicompacta.

The converse inequality $\dim \Phi \geq \text{ind}\, \Phi$ will be proved in the following section.

The inequality (3.221) easily follows from 3.21.

Indeed, for the empty set (and only for the empty set),

$$\text{ind}\, 0 = \dim 0 = -1.$$

Hence $\text{ind}\, \Phi \leq -1$ implies $\dim \Phi \leq -1$.

Let us assume that $\text{ind}\, \Phi \leq n - 1$ implies $\dim \Phi \leq n - 1$ and prove that $\text{ind}\, \Phi \leq n$ implies that $\dim \Phi \leq n$. This will also prove inequality (3.221).

But if $\text{ind}\, \Phi \leq n$, then every $x \in \Phi$ has an arbitrarily small neighborhood whose boundary has inductive dimension $\leq n - 1$. Hence the dimension of the boundary $\leq n - 1$. Hence, by Theorem 3.21, $\dim \Phi \leq n$, q.e.d.

COROLLARY TO THEOREM 3.22. *Every compactum of dimension n contains a compactum of dimension r for every $r < n$.*

REMARK. It is not yet known whether every compactum of infinite dimension contains a compactum of arbitrarily preassigned finite dimension (problem of L. A. Tumarkin).

§4. Sequences of subdivisions

§4.1. Irreducible ϵ-coverings. In this article we shall understand by covering a covering of a *given compactum* Φ.

Definition 4.11. A closed ϵ-covering

$$\alpha = \{A_1, \cdots, A_s\} \tag{4.11}$$

is said to be *irreducible* if there does not exist any closed ϵ-covering with a nerve isomorphic to a proper subcomplex of the nerve K_α of the covering α.

Irreducible ϵ-coverings possess certain remarkable properties which we shall now derive.

Definition 4.12. A closed ϵ-covering α is said to be a *special* covering if its elements are the closures of mutually disjoint open sets of the compactum Φ.

4.13. *Every irreducible ϵ-covering* (4.11) *of an n-dimensional compactum Φ has order $\leq n + 1$.*

Proof. Denote by 2σ a Lebesgue number of the irreducible covering (4.11), and let O_i be a σ-neighborhood of the set A_i in Φ. The covering

$$\omega = \{O_1, \cdots, O_s\} \tag{4.131}$$

is an open ϵ-covering of Φ similar to the covering (4.11) (see I, 8.21, 8.331).

Now denote by

$$\alpha' = \{A'_1, \cdots, A'_{s'}\}$$

any closed σ-covering of Φ of order $n + 1$. Next denote by A''_1 the union of all the A'_h which meet A_1; by A''_i the union of all the A'_h which meet A_i but do not meet any of the sets $A_1, \cdots, A_{i-1}$. Then $A''_i \subseteq O_i$. Therefore the covering α'' consisting of all the nonempty sets A''_i is an ϵ-covering whose nerve is a subcomplex of the nerve of the covering ω or, what comes to the same, of the nerve of the covering α (because the coverings α and ω are similar). Since the ϵ-covering α is irreducible, the nerve of the covering α'' is identical with the nerve of the covering α. Therefore Theorem 4.13 will follow if we prove that the order of the covering α'' does not exceed $n + 1$.

Let

$$x \in \bigcap_{j=1}^{\nu} A''_{i_j}; \qquad i_1 < i_2 < \cdots < i_\nu .$$

Since no summand A'_h appears as a summand of two different sums A''_i, there exist mutually distinct elements

$$A'_{h_j} \subseteq A''_{i_j} \qquad (1 \leq j \leq \nu)$$

of the covering α' such that

$$x \in \bigcap_{j=1}^{\nu} A'_{h_j} .$$

Hence it follows that ν does not exceed the order of the covering α', i.e., does not exceed $n + 1$, q.e.d.

Since the nerve of every closed ϵ-covering obviously contains a subcomplex which is the nerve of some irreducible ϵ-covering, it follows from 4.13 that:

4.131. The nerve of an arbitrary ϵ-covering of an n-dimensional compactum Φ contains a subcomplex which is the nerve of some ϵ-covering of the compactum Φ of order $\leq n + 1$.

4.14. *Every irreducible ϵ-covering α of a compactum Φ is similar to a special ϵ-covering α'.*

Proof. Let

$$\alpha = \{A_1, \cdots, A_s\} \tag{4.1}$$

be an irreducible ϵ-covering of Φ. Let 2σ be a Lebesgue number of the covering α; denote by O_i a σ-neighborhood of the set A_i. Then

$$\bar{\omega} = \{\bar{O}_1, \cdots, \bar{O}_s\}$$

is a closed ϵ-covering similar to the covering α. Hence for an arbitrary ϵ, Φ has an irreducible ϵ-covering whose elements are the closures of open sets.

Let us suppose that the initial covering α already has this property, i.e., that $A_i = \bar{\Gamma}_i$, $i = 1, 2, \cdots, s$, where Γ_i is open. Let us introduce the following notation:

$$A'_1 = A_1, \; A'_2 = [A_2 \setminus A'_1]^*, \; \cdots, \; A'_k = [A_k \setminus (A'_1 \cup \cdots \cup A'_{k-1})]^*,$$

where $[A_2 \setminus A'_1]^*$, etc., denotes the closure of the set included in the brackets. Obviously,

1. $A'_k \subseteq A_k$,
2. $A'_k \supseteq A_k \setminus (A_1 \cup \cdots \cup A_{k-1})$,
3. $A'_k = [\Gamma_k \setminus (A'_1 \cup \cdots \cup A'_{k-1})]^*$.

Therefore the sets A'_k form a closed ϵ-covering α' whose nerve is a subcomplex of the nerve K_α of the covering α; the nerve of α', in view of the irreducibility of α, is identical with K_α. Further, denoting by Γ'_i the set of all interior points of the set A'_i, we have, on the basis of property 3,

$$\Gamma'_k = \Gamma_k \setminus (A'_1 \cup \cdots \cup A'_{k-1}), \; A'_i = \bar{\Gamma}'_i,$$

and for $i < k$

$$\Gamma'_k \subseteq \Phi \setminus A'_i = \Phi \setminus \bar{\Gamma}'_i \subseteq \Phi \setminus \Gamma'_i,$$

which proves Theorem 4.14.

§**4.2. Subdivisions.** Let us consider anew the coverings of a compactum Φ.

4.21. A closed covering β is called a *subdivision* of the closed covering α

if every element of α is the union of elements of β and every element of β is contained in exactly one element of the covering α.

4.22. A sequence of closed coverings $\alpha_1, \alpha_2, \cdots, \alpha_m, \cdots$ is called a *chain of subdivisions* of the compactum Φ if α_m is an ϵ_m-covering of Φ, lim $\epsilon_m = 0$, and for arbitrary m the covering α_{m+1} is a subdivision of the covering α_m.

4.23. Two sequences of coverings of a compactum Φ

$$\alpha_1, \cdots, \alpha_m, \cdots$$

and

$$\beta_1, \cdots, \beta_m, \cdots$$

are said to be *similar* if the coverings α_m and β_m are similar.

The following theorem is at the basis of all the results of this section:

4.24. *Every sequence*

$$\alpha_1, \cdots, \alpha_m, \cdots, \tag{4.24}$$

where α_m is an irreducible ϵ_m-covering of a compactum Φ and lim $\epsilon_m = 0$, *contains a sequence similar to a chain of subdivisions of Φ.*

Proof. Set $m_1 = 1$ and assume that m_k has already been defined. Let us denote by m_{k+1} the first natural number such that $2\epsilon_{m_{k+1}}$ is a Lebesgue number of the covering α_{m_k}. Denote the resulting subsequence of the sequence (4.24) again by

$$\alpha_1, \cdots, \alpha_m, \cdots \tag{4.24}$$

and the corresponding ϵ by ϵ_m.

Hence (4.24) is a sequence of irreducible ϵ_m-coverings of Φ where $2\epsilon_{m+1}$ is a Lebesgue number of α_m. Hence, in particular, $2\epsilon_{m+1} < \epsilon_m$. Moreover, denoting by δ_m the maximum of the diameters of the elements of the covering α_m, we have (I, Def. 8.33)

$$\left(\textstyle\sum_{k=m+1}^{\infty} \epsilon_k\right) + \delta_m < 2\epsilon_{m+1} + \delta_m < \epsilon_m . \tag{4.241}$$

Let

$$\alpha_1 = \{A_1, \cdots, A_{i(1)}, \cdots, A_s\}.$$

Let us assume that all the elements of the covering α_m are indexed in the form

$$A_{[1]}, \cdots, A_{[s(m)]}$$

where $[1], \cdots, [s(m)]$ denote the various combinations of m indices $i(1), \cdots, i(m)$.

Let

$$A_{[1]1}, \cdots, A_{[1]s([1])}$$

be all the elements of the covering α_{m+1} which meet $A_{[1]}$. In general let

$$A_{[k]1}, \cdots, A_{[k]s([k])}$$

be all the elements of the covering α_{m+1} which meet $A_{[k]}$, but do not meet any of the sets $A_{[1]}, \cdots, A_{[k-1]}$; denote by $A^{[k]}$ the union of the elements of α_{m+1} just selected. We shall prove that $A^{[k]} \neq 0$, i.e., that there are always elements of the covering α_{m+1} which meet $A_{[k]}$ but do not meet a single $A_{[h]}$, $h < k$. Indeed, the sets $A^{[k]}$ are contained in an ϵ_{m+1}-neighborhood of the corresponding set $A_{[k]}$ and consequently form an ϵ_m-covering of Φ whose nerve is a subcomplex of the nerve K_{α_m} of the covering α_m, coinciding, in view of the irreducibility of α_m, with the whole complex K_{α_m}; whence it also follows that none of the sets $A^{[k]}$ can be empty.

Let us now consider a sequence of sets of the form

$$(4.242) \qquad A_{i(1)}, A_{i(1)i(2)}, A_{i(1)i(2)i(3)}, \cdots, A_{i(1)i(2)\cdots i(m)}, \cdots.$$

Since $\delta(A_{i(1)\cdots i(m)}) < \epsilon_1/2^m$ and $A_{i(1)\cdots i(m)} \cap A_{i(1)\cdots i(m+1)} \neq 0$, the sequence of sets (4.242) converges to a single point $x_{i(1)i(2)\cdots i(m)\cdots}$.

The set of all points $x_{i(1)\cdots i(m)h(m+1)h(m+2)\cdots}$, *where the indices* $i(1), \cdots, i(m)$ *are fixed and all the remaining indices* $h(m+1)$, $h(m+2), \cdots$ *are free to take on all the values accessible to them, is denoted by* $B_{i(1)\cdots i(m)}$.

We shall prove several properties of the sets $B_{i(1)\cdots i(m)}$.

1. It follows from the definition of these sets that

$$B_{i(1)\cdots i(m)} = \bigcup B_{i(1)\cdots i(m)i(m+1)}.$$

2. *The sets* $B_{i(1)\cdots i(m)}$ *are closed.* Indeed, let

$$(4.243) \qquad x_{i(1)\cdots i(m)h(1,m+1)h(1,m+2)\cdots}, \quad x_{i(1)\cdots i(m)h(2,m+1)h(2,m+2)\cdots}, \cdots,$$
$$x_{i(1)\cdots i(m)h(k,m+1)h(k,m+2)\cdots}, \cdots$$

be a sequence of points of the set $B_{i(1)\cdots i(m)}$.

Since each of the indices $h(k, m+1), h(k, m+2), \cdots$ can assume only a finite number of values, there exists a subsequence (4.2431) of (4.243) in which $h(k, m+1)$ takes on the same value $h(m+1)$, there exists a subsequence (4.2432) of (4.2431) in which $h(k, m+2)$ takes on a constant value $h(m+2)$, etc.

The diagonal subsequence of the sequences (4.2431), (4.2432), etc., converges to the point

$$x_{i(1)\cdots i(m)h(m+1)h(m+2)\cdots} \in B_{i(1)\cdots i(m)},$$

which proves that $B_{i(1)\cdots i(m)}$ is a compactum and is consequently a closed subset of Φ.

3. *For every given fixed m the sets $B_{i(1)\cdots i(m)}$ form a closed covering β_m of Φ.*

Indeed, let x be an arbitrary point of Φ.

Choose sets

$$(4.244) \qquad A_{i(1,1)}\,,\ A_{i(1,2)\,i(2,2)}\,,\ \cdots\,,\ A_{i(1,m)\,i(2,m)\cdots i(m,m)}\,,\ \cdots$$

to satisfy the condition that the point x be contained in each of these sets and argue as in the proof of 2. We obtain a subsequence (4.2441) of the sequence (4.244) in which the first index $i(1, m)$ assumes a constant value $i(1)$, next a subsequence (4.2442) of the sequence (4.2441) in which the second index assumes a constant value $i(2)$, etc. As a result we obtain a sequence of natural numbers

$$i(1),\ i(2),\ \cdots,\ i(m),\ \cdots$$

and the sequences (4.244), (4.2441), (4.2442), $\cdots$, (4.244m), $\cdots$ of which each (except the first) is a subsequence of the preceding, and which are such that all the elements in (4.244m) have as their first indices $i(1)i(2)\cdots i(m)$. The diagonal subsequence, as is easily seen, converges to the point $x_{i(1)\,i(2)\cdots i(m)\cdots}$, and since all the elements of this diagonal subsequence [as elements of the sequence (4.244)] contain the point x, it follows that $x = x_{i(1)\,i(2)\cdots i(m)\cdots}$, so that

$$x \in B_{i(1)\cdots i(m)}\,.$$

4. Since $B_{i(1)\cdots i(m)}$ is contained in a $(\sum_{k=m+1}^{\infty}\epsilon_k)$-, i.e., in a $2\epsilon_{m+1}$-neighborhood of the set $A_{i(1)\cdots i(m)} \in \alpha_m$, where $2\epsilon_{m+1}$ is a Lebesgue number of the covering α_m, β_m is an ϵ_m-covering of Φ, having (as a consequence of the irreducibility of α_m) the same nerve $K_{\beta_m} = K_{\alpha_m}$ as the covering α_m. Hence

The covering β_m is an irreducible ϵ_m-covering similar to the covering α_m.

Let us denote by $\Gamma_{i(1)\cdots i(m)}$ the set of those points of $B_{i(1)\cdots i(m)}$ which are not contained in any element of the covering β_m different from $B_{i(1)\cdots(m)}$. The irreducibility of the covering β_m implies that the set $\Gamma_{i(1)\cdots i(m)}$ is nonvacuous (since if $\Gamma_{i(1)\cdots i(m)}$ were empty, deleting the element $B_{i(1)\cdots i(m)}$ from β_m would yield an ϵ_m-covering β'_m whose nerve is a proper subcomplex of the complex K_{β_m}). Since $\Gamma_{i(1)\cdots i(m)}$ is the difference between Φ and the union of all the elements of the covering β_m different from $B_{i(1)\cdots i(m)}$, $\Gamma_{i(1)\cdots i(m)}$ is an open set consisting of all the points $x \in \Phi$ with the following property: for every point x represented in the form

$$x = x_{h(1)\cdots h(m)h(m+1)\cdots}\,,$$

$h(1) = i(1), \cdots, h(m) = i(m)$. Hence it follows at once that:

$$(4.245) \qquad \Gamma_{i(1)\cdots i(m)h(m+1)\cdots h(m+n)} \subseteq \Gamma_{i(1)\cdots i(m)}\,.$$

Now let x be an arbitrary point of $B_{i(1)\cdots i(m)}$ and let ϵ be an arbitrary positive number. Let us take n so large that $\epsilon_{m+n} < \epsilon$. The point x is contained in some $B_{i(1)\cdots i(m)h(m+1)\cdots h(m+n)}$ and consequently is at a distance $< \epsilon_{m+n} < \epsilon$ from $\Gamma_{i(1)\cdots i(m)h(m+1)\cdots h(m+n)}$. By (4.245) it follows that $\rho(x, \Gamma_{i(1)\cdots i(m)}) < \epsilon$. Since ϵ is arbitrary, $x \in \bar{\Gamma}_{i(1)\cdots i(m)}$. Hence $B_{i(1)\cdots i(m)} = \bar{\Gamma}_{i(1)\cdots i(m)}$. Since two different sets $\Gamma_{i(1)\cdots i(m)}$ (for the same m) are disjoint by definition, it follows that

5. *All the coverings β_m are special coverings.*

Property 5 implies that no element of the covering β_{m+1} can be contained in more than one element of the covering β_m. Since, on the other hand, $B_{i(1)\cdots i(m)i(m+1)} \subseteq B_{i(1)\cdots i(m)}$ and every $B_{i(1)\cdots i(m)}$ is the union of the $B_{i(1)\cdots i(m)i(m+1)}$ contained in it, it follows that

6. *The covering β_{m+1} is a subdivision of the covering β_m.*

The verification of all these properties of β_m proves Theorem 4.24.

Let us now assume that $\dim \Phi = r$; then all the coverings β_m have order $\leq r + 1$. Assuming that ϵ_1 is sufficiently small, we can regard the order of all the β_m as equal to $r + 1$. *The elements of the covering β_m will now be denoted by B^m_i.*

7. Each of the coverings β_m has the following property: *the intersection of any $p + 1$ elements $B^m_0, \cdots, B^m_p$ of the covering β_m, $0 \leq p \leq r$, has dimension $\leq r - p$.*

Proof. Given $\epsilon > 0$, it is required to find a closed ϵ-covering of order $\leq r - p + 1$ of the compactum $B = B^m_0 \cap \cdots \cap B^m_p$. Let us take the natural number n so large that $\epsilon_n < \epsilon$. Denote by B^n_{ik}, $k = 1, \cdots, s_i$, all the elements of the covering β_n contained in a given $B^m_i \in \beta_m$. Let us set

$$E_k = B^n_{0k} \cap B^m_1 \cap \cdots \cap B^m_p, \qquad k = 1, \cdots, s_0,$$

and prove that the sets E_k form the desired ϵ-covering of the set B. The sets E_k are closed and their diameters are $<\epsilon$ by definition. We shall prove that every point $x \in B$ is contained in at least one E_k. Since $x \in B^m_0$, x is contained in at least one B^n_{0k}. Moreover, x is contained in all of the sets B^m_i, $i = 1, \cdots, p$. Hence x is contained in some E_k. Therefore the sets E_k form a closed ϵ-covering of B. It remains to be proved that this covering has order $\leq r - p + 1$. Let the point $x \in B$ be contained in $E_1, \cdots, E_\nu$. It is necessary to prove that $\nu \leq r - p + 1$. Since $x \in B^m_i$ for all $i = 1, \cdots, p$, there is a set $B^n_{ik_i}$ containing the point x for each $i (1 \leq i \leq p)$. Moreover, $x \in B^n_{01} \cap \cdots \cap B^n_{0\nu}$. Since all the sets $B^n_{ik_i}$, $i = 1, \cdots, p$, and B^n_{0k}, $k = 1, \cdots, \nu$, are mutually distinct elements of the covering β_n having order $r + 1$, $p + \nu \leq r + 1$, i.e., $\nu \leq r - p + 1$, q.e.d.

Let us sum up the propositions proved above.

THEOREM 4.25. *Let Φ be an r-dimensional compactum. Then it is possible to construct a chain of subdivisions*

$$\beta_1, \beta_2, \cdots, \beta_m, \cdots \tag{4.25}$$

consisting of coverings β_m of Φ with the following properties:

a) *Every β_m is a special irreducible ϵ_m-covering of order $r + 1$.*

b) *The intersection of any $p + 1$, $0 \leq p \leq r$, elements of the covering β_m has dimension $\leq r - p$.*

We shall deduce from this the following important proposition.

4.26. *Every point of an r-dimensional compactum Φ has an arbitrarily small neighborhood whose boundary has dimension $\leq r - 1$.*

Proof. Let $\epsilon > 0$ be given. Take an m such that $2\epsilon_m < \epsilon$. For a given point $x \in \Phi$ define the neighborhood $O_m x$ as the difference between Φ and the union of all the elements of the covering β_m which do not contain the point x:

$$O_m x = \Phi \setminus \cup B^m{}_h, \qquad x \notin B^m{}_h.$$

The closure of the neighborhood $O_m x$ is contained in the union of all the $B^m{}_i \in \beta_m$ which contain x. Hence its diameter is $\leq 2\epsilon_m < \epsilon$. The boundary of the neighborhood $O_m x$ is, on the one hand, contained in the union of all the $B^m{}_i \in \beta_m$ containing x and, on the other hand, in the union of all $B^m{}_h \in \beta_m$ which do not contain x. In other words, the boundary of $O_m x$ is contained in the union of all sets of the form $B^m{}_i \cap B^m{}_h$, where $B^m{}_i \in \beta_m$ contains x, but $B^m{}_h \in \beta_m$ does not contain x. Since each of the sets $B^m{}_i \cap B^m{}_h$ has, by virtue of 4.25, dimension $\leq n - 1$, it follows from Theorem 3.1 that the dimension of their union, and hence also the dimension of the boundary of the neighborhood $O_m x$, is $\leq n - 1$, q.e.d.

It is easy to infer from Theorem 4.26 that:

4.260. *Let A be a closed set of an n-dimensional compactum Φ. For every neighborhood OA of the set $A \subset \Phi$, there exists a neighborhood O_1A of A contained in OA whose boundary has dimension $\leq n - 1$ (and is also contained in OA).*

Proof. For every $x \in A$ let Ox be a neighborhood of x satisfying the conditions

$$\bar{O}x \subseteq OA, \qquad \dim (\bar{O}x \setminus Ox) \leq n - 1.$$

Since A is a compactum, we can choose a finite number of these neighborhoods $O_1 = Ox_1, \cdots, O_s = Ox_s$, covering the whole set A. The union of these neighborhoods is a neighborhood O_1A of A, where the boundary of O_1A is contained, as is easily seen, in the union of the boundaries of the neighborhoods $O_1, \cdots O_s$. Hence the boundary of O_1A has dimension $\leq n - 1$. Since $\bar{O}_1A \subseteq OA$, the proof is complete.

Let us derive the inequality

$$\text{ind}\,\Phi \leq \dim \Phi \tag{4.261}$$

from Theorem 4.26. This will conclude the proof of Theorem 3.22.

If $\dim \Phi = -1$ (i.e., $\dim \Phi \leq -1$), $\Phi = 0$, so that $\text{ind}\,\Phi \leq -1$. Let us assume as proved that $\dim \Phi \leq n - 1$ implies $\text{ind}\,\Phi \leq n - 1$; we shall prove that $\dim \Phi \leq n$ implies $\text{ind}\,\Phi \leq n$; this will also prove (4.261).

If $\dim \Phi \leq n$, then, by Theorem 4.26, for each $x \in \Phi$ there exists an arbitrarily small neighborhood whose boundary has dimension $\leq n - 1$. Consequently, by hypothesis, the inductive dimension is also $\leq n - 1$. But then, by the definition of inductive dimension, $\text{ind}\,\Phi \leq n$, q.e.d.

Let us derive from Theorem 4.25 yet another important proposition.

Let us say that a mapping C of a set X onto a set Y has *order* n if the inverse image $C^{-1}(y)$ consists of no more than n elements of X for every $y \in Y$ and if $C^{-1}(y_0)$ contains exactly n elements of X for at least one $y_0 \in Y$.

Theorem 4.27. *Every compactum Φ of dimension n without isolated points can be represented as the image of the Cantor perfect set under a continuous mapping of order $n + 1$; conversely, every compactum which is the image of the Cantor perfect set under a continuous mapping of order $n + 1$ has dimension $\leq n$ and does not contain isolated points.*

We shall prove the second part of the theorem first. Let the compactum Φ be the image of the Cantor perfect set P under a continuous mapping C of order $n + 1$. Let us divide P into two pieces (by a piece of the Cantor perfect set we mean its intersection with an arbitrary segment of the real line whose endpoints do not belong to P; we shall consider only non-empty pieces) P_1 and P_2 of diameter $\frac{1}{3}$, then each of these into two pieces P_{11} and P_{12} of diameter $\frac{1}{9}$, etc.

The images $B_{i(1)\cdots i(m)} = CP_{i(1)\cdots i(m)}$ of the pieces $P_{i(1)\cdots i(m)}$ (for a given m) form an ϵ_m-covering β_m of Φ, where $\lim \epsilon_m = 0$. If, for a given m, a point $x \in \Phi$ is contained in ν distinct elements $B_{i(1)\cdots i(m)}$ of β_m, then ν distinct pieces $P_{i(1)\cdots i(m)}$ contain a point of the inverse image $C^{-1}(x)$. Since C is a mapping of order $n + 1$, $\nu \leq n + 1$, i.e., the order of each of the coverings β_m does not exceed $n + 1$. Consequently

$$\dim \Phi \leq n.$$

If Φ were to contain an isolated point x, then $C^{-1}x$ would be an open set in P and since there are no finite (even countable) open sets in P, C could not be of finite order.

This proves the second half of Theorem 4.27.

Let us prove the first half of Theorem 4.27. Let Φ be an n-dimensional compactum without isolated points.

Let us construct a chain of subdivisions

$$\beta_1, \beta_2, \cdots, \beta_m, \cdots \tag{4.27}$$

for Φ satisfying all the conditions of Theorem 4.25 and, moreover, the following additional condition: the covering β_1 contains at least two elements and for arbitrary m the diameter of each element of the covering β_{m+1} is less than one half the minimum of the diameters of the elements of the covering β_m. Since all the β_m are special coverings and Φ has no isolated points, β_m has no degenerate elements, so that the diameters of all the $B^m{}_{i(1)\cdots i(m)} \in \beta_m$ are different from zero and the supplementary condition can be fulfilled.

This condition implies that each $B^m{}_{i(1)\cdots i(m)} \in \beta_m$ contains at least two sets $B^{m+1}{}_{i(1)\cdots i(m)i(m+1)} \in \beta_{m+1}$.

Let us now divide the Cantor perfect set P into the same number of mutually disjoint pieces $P_{i(1)}$ of diameter $\leq \frac{1}{2}$ as the number of elements $B_{i(1)}$ in the covering β_1. Since the diameter of P is equal to 1 and the number of necessary pieces is ≥ 2, the requirement that $\delta(P_{i(1)}) \leq \frac{1}{2}$ is fulfilled.

Suppose that for a given m the Cantor set P has been divided into mutually disjoint pieces $P_{i(1)\cdots i(m)}$ of diameter $\leq 1/2^m$ corresponding $(1-1)$ to the elements of the covering β_m. Let us divide each piece $P_{i(1)\cdots i(m)}$ into the same number of pieces of diameter $\leq 1/2^{m+1}$ as the number of elements in the covering β_{m+1} contained in $B_{i(1)\cdots i(m)}$. Hence the pieces $P_{i(1)\cdots i(m)}$ are constructed for any m. Assign to each point

$$\xi = P_{i(1)} \cap P_{i(1)i(2)} \cap \cdots \cap P_{i(1)i(2)\cdots i(m)} \cap \cdots$$

of the set P the point

$$x = C(\xi) = B_{i(1)} \cap B_{i(1)i(2)} \cap \cdots \cap B_{i(1)i(2)\cdots i(m)} \cap \cdots$$

of the compactum Φ. It is easy to see that this assignment is a continuous mapping C of P onto Φ. If C maps ν distinct points $\xi_1, \cdots, \xi_\nu$ of P onto the same point of Φ, let m be such that no two of the points $\xi_1, \cdots, \xi_\nu$ are contained in the same set $P_{i(1)\cdots i(m)}$. Then the point $x = C(\xi_1) = \cdots = C(\xi_\nu)$ is contained in ν elements of the covering β_m. Since all the coverings β_m have order $\leq n + 1$, $\nu \leq n + 1$, i.e., the order of the mapping C does not exceed $n + 1$, q.e.d.

EXERCISE. Prove the following theorem:

Every compactum Φ of dimension $\leq n$ is the image of a closed subset of the Cantor perfect set under a continuous mapping of order $\leq n + 1$; conversely, all the compacta which are images of closed subsets of the Cantor perfect set under continuous mappings of order $\leq n + 1$, have dimension $\leq n$.

In other words, *the dimension of a compactum Φ can be defined as the*

least number n satisfying the following condition: there exists a continuous mapping of order $n + 1$ of a closed subset of the Cantor perfect set onto the compactum Φ.

§5. Some applications to topological manifolds and polyhedra

§5.1. The case of topological manifolds (in particular R^n and S^n). Let M^n be a closed topological n-manifold. We know that $\dim M^n = n$, so that $\operatorname{ind} M^n = n$. Therefore, there exists a point $x \in M^n$ for which $\operatorname{ind}_x M^n = n$. But since all the points $x \in M^n$ have homeomorphic neighborhoods, it follows that for an *arbitrary* $x \in M^n$, $\operatorname{ind}_x M^n = n$. In particular,

$$\operatorname{ind}_x S^n = n$$

and

$$\operatorname{ind}_x R^n = n$$

at an arbitrary point $x \in S^n$, $x \in R^n$, respectively. Hence

5.11. *Every topological n-manifold has inductive dimension n at each of its points.*

This implies the following theorem:

5.120. *No compactum of dimension $\leq n - 2$ separates the space R^n.*

Proof. Let us assume that a compactum Φ of dimension $\leq n - 2$ separates R^n.

Let Γ be a bounded component of the open set $R^n \setminus \Phi$ and p any point of the domain Γ. Let $\bar{E}^n$ be a solid sphere with center p and of sufficiently large radius to contain Φ in its interior. Let $\bar{e}^n$ be a solid sphere of radius ϵ, $\epsilon > 0$ arbitrary, with center at the origin o of R^n.

Let us map the sphere $\bar{E}^n$ onto the sphere $\bar{e}^n$ by means of a similitude. Then the domain Γ is mapped into a domain γ containing the point o and the compactum Φ is mapped into a compactum φ of dimension $\leq n - 2$. Here γ and φ are contained in the interior of the sphere $\bar{e}^n$ and the boundary of the domain γ is contained in φ. Consequently the boundary of γ has dimension $\leq n - 2$. Since γ is a neighborhood of the point o having diameter $< 2\epsilon$, and ϵ is arbitrary, $\operatorname{ind}_o R^n \leq n - 1$, which contradicts Theorem 5.11. This contradiction proves Theorem 5.120.

Theorem 5.120 in turn implies

5.12. *A closed solid n-sphere $\bar{E}^n$ is not separated by any compactum of dimension $\leq n - 2$.*

Indeed, suppose that the closed solid n-sphere $\bar{E}^n$ is separated by a compactum Φ of dimension $\leq n - 2$. With no restriction on generality, we may assume that Φ does not contain the center o of the sphere.

Our assumptions imply that

$$\bar{E}^n = A \cup \Phi \cup D,$$

where A, Φ, D are mutually disjoint and A and D are nonempty open subsets of $\bar{E}^n$. For definiteness let $o \in A$. An inversion maps the sphere $S^n = \bar{E}^n \setminus E^n$ onto itself, the open sets $A \setminus o$ and D into A_1 and D_1, respectively, and the closed set Φ into a closed set Φ_1; here

$$R^n = (A \cup A_1) \cup (\Phi \cup \Phi_1) \cup (D \cup D_1).$$

It is easy to see that

$$A \cup A_1, \qquad \Phi \cup \Phi_1, \qquad D \cup D_1$$

are mutually disjoint and that $A \cup A_1$, $D \cup D_1$ are open and nonempty so that the compactum $\Phi \cup \Phi_1$ separates the space R^n. This is impossible since $\dim \Phi_1 = \dim \Phi \leq n - 2$ and, by Theorem 3.1, $\dim (\Phi \cup \Phi_1)$ is also $\leq n - 2$. This proves Theorem 5.12.

5.13. *No n-dimensional topological manifold M^n is separated by a compactum of dimension $\leq n - 2$.*

Proof. For every point $p \in M^n$ let $U(p)$ be a neighborhood of p homeomorphic to R^n. Let C be a homeomorphism of R^n onto $U(p)$ and let C map a closed solid sphere $\bar{E}^n \subset R^n$ onto $\bar{V}(p) \subset U(p)$, where $V(p) = C(E^n)$. Hence we obtain for each point $p \in M^n$ a definite neighborhood $V(p)$ homeomorphic to R^n, where $\bar{V}(p)$ is homeomorphic to a closed solid sphere. We can choose a countable, and in the case of a closed manifold M^n even a finite, set of these neighborhoods $V(p)$ whose union is M^n.

Let these neighborhoods $V(p)$ be

$$V_1, V_2, \cdots, V_s, \cdots. \tag{5.13}$$

Since M^n is connected, the system of sets (5.13) is chained by I, Theorem 3.18.

Now let $\Phi \subset M^n$ be a compactum of dimension $\leq n - 2$. Each of the sets $\Phi \cap \bar{V}_i$ is nowhere dense in $\bar{V}_i$ by Theorem 1.25. Consequently, the system of sets

$$\bar{V}_1 \setminus \Phi, \ \bar{V}_2 \setminus \Phi, \cdots, \bar{V}_s \setminus \Phi, \cdots$$

is also chained. Each of the sets $\bar{V}_i \setminus \Phi$ is connected by 5.12 so that by I, Theorem 3.15, the union of the sets $\bar{V}_i \setminus \Phi$, i.e., the set $M^n \setminus \Phi$, is connected.

In particular,

5.131. *No domain of the space R^n is separated by a compactum of dimension $\leq n - 2$.*

§5.2. Strong connectedness.

DEFINITION 5.21. A compactum Φ of dimension n is said to be *strongly connected* if no closed set of dimension $\leq n - 2$ separates Φ (strongly connected n-dimensional compacta were introduced by Urysohn under the name of Cantorian manifolds).

From this definition and Theorem 3.22 it follows immediately that:

5.22. *If Φ is a strongly connected n-dimensional compactum, then* $\mathrm{ind}_x\Phi = n$ *for every point* $x \in \Phi$.

In view of the results of the preceding article, all closed topological manifolds such as the closed solid n-sphere and the compacta homeomorphic to it are examples of strongly connected compacta. It is left to the reader to prove that *the closure of every domain of a given closed topological manifold is a strongly connected compactum.*

REMARK. The following theorem whose proof will not be given in this book is valid:

THEOREM of Hurewicz-Tumarkin. *Every n-dimensional compactum contains a strongly connected n-dimensional compactum.*

Let us now go on to consider strongly connected polyhedra.

DEFINITION 5.23. A finite sequence of n-simplexes

$$T^n{}_1, \cdots, T^n{}_s$$

of a given simplicial complex K is called a *chain of simplexes of* K (more precisely: a chain connecting the simplexes $T^n{}_1$ and $T^n{}_s$ in K), if the simplexes $T^n{}_i$ and $T^n{}_{i+1}$, $i = 1, 2, \cdots, s - 1$, have a common $(n - 1)$-face in K.

DEFINITION 5.24. A simplicial n-complex K is said to be *pure* if each of its simplexes is a face of an n-simplex of K. A pure n-complex K is said to be *strongly connected* if every two n-simplexes of K can be connected by a chain of n-simplexes in K.

5.251. *If a triangulation K is a strongly connected n-complex, $\| K \|$ is a strongly connected n-dimensional polyhedron.*

Proof. Let $T^n{}_1, \cdots, T^n{}_s$ be all the n-simplexes of the complex K. Then since K is pure,

$$\| K \| = \overline{T}^n{}_1 \cup \cdots \cup \overline{T}^n{}_s . \tag{5.251}$$

Let $\Phi \subset \| K \|$ be an arbitrary closed set of dimension $n - 2$. If T^{n-1} is an arbitrary $(n - 1)$-simplex of K,

$$\overline{T}^{n-1} \setminus \Phi \neq 0;$$

it easily follows from this and the strong connectedness of K that the system of sets

$$\overline{T}^n{}_1 \setminus \Phi, \cdots, \overline{T}^n{}_s \setminus \Phi$$

is chained. Then (by I, 3.15)

$$\| K \| \setminus \Phi = (\overline{T}^n{}_1 \setminus \Phi) \cup \cdots \cup (\overline{T}^n{}_s \setminus \Phi)$$

is connected. This proves the assertion.

5.252. *Every triangulation K of a strongly connected n-dimensional polyhedron Φ is a strongly connected complex.*

Indeed, from 5.22 it follows first that K is a pure complex. Suppose that K is not strongly connected. Then there exist two n-simplexes $T^n_1 \in K$, $T^n_2 \in K$ which cannot be connected by any chain in K. Denote by Q_1 the set of all n-simplexes of K which can be connected by chains with T^n_1; let Q_2 be the set of all the remaining n-simplexes of K; both sets Q_1 and Q_2 are nonempty (since $T^n_1 \in Q_1$, $T^n_2 \in Q_2$). The combinatorial closures

$$K_1 = | Q_1 | \quad \text{and} \quad K_2 = | Q_2 |$$

are closed subcomplexes of the triangulation K whose intersection K_0 has dimension $\leq n - 2$. Since $K = K_1 \cup K_2$,

$$\| K \| = \| K_1 \| \cup \| K_2 \|$$

and

$$\| K \| \setminus \| K_0 \| = (\| K_1 \| \setminus \| K_0 \|) \cup (\| K_2 \| \setminus \| K_0 \|).$$

Since $\| K_1 \| \setminus \| K_0 \|$ and $\| K_2 \| \setminus \| K_0 \|$ are disjoint nonempty open subsets of the polyhedron $\| K \|$, $\| K \|$ is separated by the polyhedron $\| K_0 \|$ of dimension $\leq n - 2$, which contradicts the strong connectedness of $\| K \|$.

Hence

5.25. *Strongly connected polyhedra can be defined as the bodies of strongly connected triangulations; if one triangulation of a polyhedron Φ is strongly connected, then every triangulation of every polyhedron homeomorphic to Φ has the same property.*

Appendix 1

N-DIMENSIONAL ANALYTIC GEOMETRY

Preliminary remarks. Certain definitions and theorems of n-dimensional analytic geometry applied in the book are collected in this appendix. It is assumed that the reader's knowledge of this subject approximates the contents of the book of Schreier and Sperner [S–S]. Many of the proofs are accordingly left to the reader.

We shall use vector notation for calculations involving points of n-space.

If a and b are points of the Euclidean n-space R^n with coordinates α_1, $\alpha_2, \cdots, \alpha_n$ and $\beta_1, \beta_2, \cdots, \beta_n$, respectively, then $a + b$ is the point with coordinates

$$\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_n + \beta_n .$$

Furthermore, λa is the point with coordinates

$$\lambda\alpha_1, \lambda\alpha_2, \cdots, \lambda\alpha_n .$$

We shall denote the number $[\sum_{i=1}^{n} \alpha_i^2]^{\frac{1}{2}}$ by $|a|$. Then the distance between the points a and b is

$$\rho(a, b) = |a - b| .$$

The triangle axiom may now be written as

$$|c - a| \leq |c - b| + |b - a| ,$$

or, replacing $c - b$ by a and $b - a$ by b,

$$|a + b| \leq |a| + |b| .$$

Moreover,

$$|\lambda a| = |\lambda| \cdot |a| .$$

This relation combined with the triangle inequality yields

$$\left|\sum \lambda_i a_i\right| \leq \sum |\lambda_i| \cdot |a_i| .$$

§1. The space R^n and planes in R^n

§1.1. In this book the Euclidean n-space is always denoted by R^n. An r-dimensional subspace, $0 \leq r \leq n$, of R^n is denoted by R^r or sometimes by X^r or Y^r and is called an r-dimensional (hyper)plane of R^n or simply an r-plane. R^n is itself the unique n-plane of R^n.

Every $(n - 1)$-plane R^{n-1} of R^n divides R^n into two open half-spaces $H^n{}_1$ and $H^n{}_2$: if the plane R^{n-1} is defined by the equation

$$a_1x_1 + \cdots + a_nx_n = 0,$$

the open half-spaces H^n_1 and H^n_2 are defined by the inequalities $a_1x_1 + \cdots + a_nx_n > 0$ and $a_1x_1 + \cdots + a_nx_n < 0$. Two points of R^n are said to lie *on one side* of the plane R^{n-1} if they are in the same half-space determined by R^{n-1} and *on different sides* of R^{n-1} if one point is in one half-space and the other in the second half-space.

The closure of an open half-space H^n_i, $i = 1, 2$, i.e.,

$$\bar{H}^n_i = H^n_i \cup R^{n-1}, \qquad i = 1, 2$$

is called a *closed half-space.*

§1.2. Linear independence of points. Barycentric coordinates. The points $a_0, a_1, \cdots, a_r$ of R^n are said to be *linearly independent* if they are not contained in any plane of dimension $<r$.

Accordingly, every $n + 2$ or more points of R^n are linearly dependent.

The points a_i, $i = 0, 1, \cdots, r$, with coordinates $x^{(1)}_i, x^{(2)}_i, \cdots, x^{(n)}_i$ are linearly independent if, and only if, the matrix

$$\begin{pmatrix} x^{(1)}_0 & \cdots & x^{(n)}_0 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x^{(1)}_r & \cdots & x^{(n)}_r & 1 \end{pmatrix}$$

has rank r. This may be reformulated as follows: if

$$\lambda_0 a_0 + \cdots + \lambda_r a_r = 0$$

and

$$\lambda_0 + \lambda_1 + \cdots + \lambda_r = 0$$

then

$$\lambda_0 = \lambda_1 = \cdots = \lambda_r = 0.$$

If $a_0, a_1, \cdots, a_r$ are $r + 1$ linearly independent points of R^n, there is exactly one r-plane, the plane $R(a_0, \cdots, a_r)$ spanned by the points $a_0, \cdots, a_r$, passing through these points. It is contained in every plane R^s containing the points $a_0, a_1, \cdots, a_r$.

Barycentric coordinates. The plane $R(a_0, \cdots, a_r)$ consists of those points of R^n which can be represented in the form

$$a = \mu_0 a_0 + \mu_1 a_1 + \cdots + \mu_r a_r, \tag{1.2}$$

with the supplementary condition that

$$\mu_0 + \mu_1 + \cdots + \mu_r = 1; \tag{1.21}$$

on the other hand, the numbers $\mu_0, \cdots, \mu_r$ are uniquely determined by the point a and the relations (1.2), (1.21); they are called the *barycentric*

coordinates of the point a in the coordinate system $a_0, \cdots, a_r$. This terminology, introduced by F. A. Möbius, has the following basis: let us call a point of R^n to which a real number σ, the mass or weight of the point, is assigned a mass point. If mass points a_i with weights σ_i are given and $\mu_i = \sigma_i/(\sigma_0 + \cdots + \sigma_r)$, then the point (1.2) is, by definition, the center of gravity or centroid of this mass distribution.

The numbers σ_i may or may not be positive. They are arbitrary real numbers such that

$$\sigma_0 + \cdots + \sigma_r \neq 0.$$

§1.3. Theorems on intersections and linear closures. By the intersection of two planes R^r and R^s of R^n we understand, as usual, the set of points common to both planes; the plane spanned by R^r and R^s, or the *linear closure* of the two planes, is the plane of least dimension which contains both R^r and R^s; it is obviously unique.

The following theorems are easily proved:

1.31. *The intersection of two planes R^r and R^s is either empty or is a plane R^d, $d \geq r + s - n$.*

1.32. *The dimension h of the linear closure of two planes R^r and R^s satisfies the inequality $h \leq r + s + 1$.*

1.33. *If the intersection of two planes R^r and R^s is nonempty, then $d + h = r + s$.*

Since $h \leq n$, 1.31 is contained in 1.33.

§1.4. General position.

DEFINITION. A set of points of R^n is said to be in *general position* if every r points, $r \leq n + 1$, of the set are linearly independent.

The theorems of 1.3 imply

1.41. *Let the points $a_0, \cdots, a_r, b_0, \cdots, b_s$ be in general position and let $r \leq n$, $s \leq n$. Then the following assertions hold for the intersection and linear closure of the planes $R^r = R(a_0, \cdots, a_r)$ and $R^s = R(b_0, \cdots, b_s)$:*

If $r + s < n$, the linear closure has dimension $r + s + 1$ and the intersection is empty.

If $r + s \geq n$, the linear closure has dimension n (hence it coincides with the space R^n) and the intersection is either empty or has dimension $r + s - n$.

Every finite set of points can be brought into general position by means of an arbitrarily small displacement, i.e.,

1.42. *Let $\{a_0, \cdots, a_s\}$ be an arbitrary finite set of points in R^n and ϵ a positive number. Then there exists a set of points $\{a'_0, \cdots, a'_s\}$ in general position such that*

$$\rho(a'_i, a_i) < \epsilon, \qquad i = 0, 1, \cdots, s.$$

Proof. Set $a'_0 = a_0$ and suppose that points $a'_i \in S(a_i, \epsilon)$, $i \leq m \leq s - 1$, have already been found to satisfy the condition that $a'_0, \cdots, a'_m$ are in general position. Then choose a point $a'_{m+1} \in S(a_{m+1}, \epsilon)$ not contained in any of the planes $R'(a'_{i_0}, \cdots, a'_{i_r})$, where $i_0, \cdots, i_r$ do not exceed m and $r < n$. If $m = s - 1$, the theorem is proved.

If the points $a_0, \cdots, a_r$ are linearly independent, i.e., are not contained in any plane of dimension $<r$, the same is true for any points $a'_0, \cdots, a'_r$ which are at a sufficiently small distance from $a_0, \cdots, a_r$, respectively. From this simple remark it follows that:

1.43. *If the points $a_0, \cdots, a_k$ are in general position, there exists a $\delta > 0$ such that any points $a'_0, \cdots, a'_k$ for which $\rho(a_i, a'_i) < \delta$, $i = 0, 1, \cdots, k$, are also in general position.*

§1.5. Affine mappings. Concerning affine mappings of R^n into itself see e.g., Schreier and Sperner [S–S, §13]. Of the properties of affine mappings we recall first the *invariance of the centroid* (easily established by simple calculations): if C is an affine mapping of R^n into itself, and if a is the centroid of points a_k, $k = 0, 1, \cdots, r$, with weights m_k, then $C(a)$ is the centroid of the mass points $C(a_k)$ with the same weights m_k.

REMARK. It is assumed here that if $C(a_h) = C(a_k) = b$, then the weight $m_h + m_k$ is located at the point b.

The invariance of the centroid implies that:

1.51. *If $a_0, \cdots, a_n$ is a set of $n + 1$ linearly independent points of R^n and $b_0, \cdots, b_n$ is any set of $n + 1$ points in R^n, there exists a unique affine mapping C of R^n onto a linear subspace of R^n which takes the points $a_0, \cdots$, a_n*, into *$b_0, \cdots, b_n$, respectively*. The mapping C is obtained by assigning to an arbitrary point a with barycentric coordinates $\mu_0, \cdots, \mu_n$ (with respect to $a_0, \cdots, a_n$) the centroid of the weights $\mu_0, \cdots, \mu_n$ located at the points $b_0, \cdots, b_n$.

An affine mapping is said to be *non-singular* if it is (1–1). Otherwise it is *singular*. An affine mapping is *onto* (R^n) if, and only if, it is non-singular. Non-singular mappings may be characterized by the fact that they map every linearly independent system of $n + 1$ points of R^n onto a linearly independent system, also consisting of $n + 1$ points.

This condition is expressed analytically as:

An affine mapping C defined by the relations

$$x'_i = u^{(1)}{}_i x_1 + \cdots + u^{(n)}{}_i x_n + u_i, \qquad i = 1, \cdots, n, \tag{1.51}$$

(where x_i and x'_i are the coordinates of the points of the inverse images and images, respectively) is non-singular if, and only if, the determinant of the mapping is different from zero:

$$\det C = \begin{vmatrix} u^{(1)}{}_1 & \cdots & u^{(n)}{}_1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ u^{(1)}{}_n & \cdots & u^{(n)}{}_n \end{vmatrix} \neq 0.$$

An affine mapping is said to *positive* if its determinant is positive, *negative* if its determinant is negative. This definition is legitimate, since the sign of the determinant is independent of the choice of the coordinate system.

Let C_θ, $0 \leq \theta \leq 1$, be a family of non-singular affine mappings of R^n onto itself depending on the parameter θ (I, 7.4). All the mappings C_θ, in particular C_0 and C_1, have the same sign since the determinant of C_θ is continuous in θ and hence, never assuming the value zero, cannot change in sign.

The identity mapping C_0 of R^n onto itself obviously has determinant $+1$ in every system of coordinates; hence every affine mapping which can be transformed into the identity mapping by means of a continuous deformation C_θ (where all the C_θ are non-singular affine mappings of R^n onto itself) is positive. As an example of a negative affine mapping of R^n onto itself we shall consider a *symmetry relative to an* $(n - 1)$- *plane* $R^{n-1} \subset R^n$.

If the coordinate system is chosen so that the plane R^{n-1} is the coordinate plane $x_n = 0$, the mapping can be defined by the equations

$$x'_i = x_i, \qquad i = 1, \cdots, (n-1),$$
$$x'_n = -x_n,$$

whence it is at once clear that the determinant of the mapping in the chosen coordinate system is -1. Hence every affine mapping which can be continuously deformed into a symmetry relative to an $(n - 1)$-plane by a set of non-singular affine mappings of R^n onto itself is negative. We shall make use of this example in the proof of the following theorem which is required in Chapter VII:

1.52. *Let* $e_1, \cdots, e_n$ *be a system of* n *linearly independent points of an* $(n - 1)$*-plane* $R^{n-1} \subset R^n$, *and let* e'_0 *and* e''_0 *be two points in the complement of* R^{n-1}. *Then an affine mapping* C_0 *which takes the linearly independent points* $e'_0, e_1, \cdots, e_n$ *into* $e''_0, e_1, \cdots, e_n$, *respectively, is positive if* e'_0 *and* e''_0 *lie on one side of* R^{n-1} *and negative in the contrary case.*

Theorem 1.52 is a special case of the following proposition (which is utilized in Chapter X):

1.521. *Let* C *be an affine mapping of* R^n *onto itself which maps some* $(n - 1)$*-plane* R^{n-1} *onto itself. The affine mapping of* R^{n-1} *defined by* C *is denoted by* C'.

Let ϵ be a number equal to $+1$ if C maps each of the two half-spaces into which R^{n-1} divides R^n onto itself and equal to -1 if C maps each of these half-spaces onto the other. Then the signs of C and C' satisfy the relation

$$\operatorname{sgn} C \cdot \operatorname{sgn} C' = \epsilon$$

(in particular, if $\operatorname{sgn} C' = +1$, $\operatorname{sgn} C = \epsilon$, whence 1.52 also follows).

Proof of 1.521. The theorem is obvious for $n = 1$; let $n \geq 2$. Let $x_1, \cdots, x_n$ be a coordinate system in R^n such that R^{n-1} is the coordinate plane $x_1 = 0$. In this coordinate system we may write C in the form

$$x'_i = \textstyle\sum_{k=1}^{n} a_{ik}x_k, \qquad i = 1, 2, \cdots, n.$$

Since R^{n-1} is transformed into itself, $x'_1 = 0$ for $x_1 = 0$ and arbitrary $x_2, \cdots, x_n$, i.e., $a_{12}x_2 + \cdots + a_{1n}x_n = 0$ for arbitrary $x_2, \cdots, x_n$ so that

$$a_{12} = \cdots = a_{1n} = 0.$$

In other words, the mapping C has the form

$$\begin{aligned} x'_1 &= a_{11}x_1, \\ x'_2 &= a_{21}x_1 + \cdots + a_{2n}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x'_n &= a_{n1}x_1 + \cdots + a_{nn}x_n, \end{aligned}$$

whence it follows that

$$\det C = a_{11} \det C'.$$

But a_{11} has the same sign as ϵ so that $\operatorname{sgn} C = \epsilon \operatorname{sgn} C'$, q.e.d.

An affine mapping which transposes any two of $n + 1$ linearly independent points $e_0, \cdots, e_n$, i.e., which takes the points $e_0, \cdots, e_i, \cdots, e_k, \cdots, e_n$ into $e_0, \cdots, e_k, \cdots, e_i, \cdots, e_n$, respectively, is a negative mapping. To show this it suffices to take an affine system of coordinates in R^n whose unit vectors are $e_0e_1, \cdots, e_0e_n$ with origin e_0: the determinant of the mapping in this coordinate system is obviously -1.

It follows from this remark that:

1.53. *If the points $e_0, \cdots, e_n$ are linearly independent in R^n, the sign of the affine mapping realizing a given permutation*

$$\begin{pmatrix} e_0 & \cdots & e_n \\ e_{i_0} & \cdots & e_{i_n} \end{pmatrix}$$

of these points is the same as the sign of the permutation.

§1.6. Definition of an affine mapping of R^n by affine mappings of two planes X^p and Y^q, $p + q = n$.

Let $p + q = n$, and let X^p, Y^q be two planes in R^n intersecting in a single point o and such that the linear closure of the planes coincides with all of R^n.

Let C_1 and C_2 be affine mappings of X^p and Y^q, respectively, onto themselves such that $C_1(o) = C_2(o) = o$. In these conditions, *there exists a unique affine mapping C of R^n onto itself* coinciding on X^p and Y^q with C_1 and C_2, respectively. The mapping C is called the induced mapping of C_1 and C_2 or simply the *extension* of these mappings over R^n.

Proof. Choose a system of coordinates with origin o in R^n such that the first p unit vectors of the coordinate system are in X^p and the rest in Y^q. If C_1, C_2, in the coordinates $x_1, \cdots, x_p; x_{p+1}, \cdots, x_n$, are given by the equations

$$\begin{array}{ll} x'_1 = a_{11}x_1 + \cdots + a_{1p}x_p & x'_{p+1} = a_{p+1,p+1}x_{p+1} + \cdots + a_{p+1,n}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ x'_p = a_{p1}x_1 + \cdots + a_{pp}x_p & x'_n = a_{n,p+1}x_{p+1} + \cdots + a_{nn}x_n \end{array}$$

respectively, then the desired mapping C in the coordinates $x_1, \cdots, x_p, \cdots, x_n$ is written as

$$\begin{array}{lll} x'_1 = & a_{11}x_1 + \cdots + a_{1p}x_p & \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \\ x'_p = & a_{p1}x_1 + \cdots + a_{pp}x_p & \\ x'_{p+1} = & & a_{p+1,p+1}x_{p+1} + \cdots + a_{p+1,n}x_n \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ x'_n = & & a_{n,p+1}x_{p+1} + \cdots + a_{nn}x_n . \end{array}$$

COROLLARY.

$$\det C = \det C_1 \cdot \det C_2 . \tag{1.61}$$

§2. Convex sets

§2.1. Definition of convex sets. The straight line defined by two points a, b is the set of all points of the form $\lambda a + \mu b$, where $\lambda + \mu = 1$ (see 1.2). The subset of this straight line defined by the conditions $\lambda \geq 0$, $\mu \geq 0$; $\lambda > 0$, $\mu > 0$, respectively, is called the closed segment $[ab]$ or the open segment (ab), respectively.

DEFINITION 2.11. A set M of points of the space is said to be *convex* if it contains with every two of its points a and b the whole segment $[ab]$.

The simplest examples of convex sets are: the whole space R^n and its planes (of arbitrary dimension), half-spaces, segments, sets consisting of one point, the empty set.

DEFINITION 2.12. *A closed bounded convex subset of R^n containing an interior point (relative to R^n) is called an n-dimensional convex body.*

§2.2. Simplest properties of convex sets.

2.21. *Every convex set is connected.*

In fact, two arbitrary points p and q of a convex set M are contained in a connected subset of M (the segment $[pq]$), i.e., M is connected. It follows immediately from Def. 2.11 that:

2.22. *The intersection of an arbitrary (finite or infinite) set of convex sets is convex.*

Furthermore,

2.23. *If M is convex and $\delta > 0$, then $S(M, \delta)$ is convex.*

Proof. Let b_1, b_2 be points of the neighborhood $S(M, \delta)$ and let b be a point of the segment $[b_1b_2]$; it is required to show that b is contained in $S(M, \delta)$.

There exist points a_1, a_2 in M such that $| b_1 - a_1 | < \delta$, $| b_2 - a_2 | < \delta$. Then since b is a point of the segment $[b_1b_2]$,

$$b = \lambda b_1 + \mu b_2, \qquad \lambda \geq 0, \qquad \mu \geq 0, \qquad \lambda + \mu = 1.$$

If the point $\lambda a_1 + \mu a_2$ of the segment $[a_1a_2]$ is denoted by a, then

$$b - a = \lambda(b_1 - a_1) + \mu(b_2 - a_2),$$

whence

$$| b - a | \leq \lambda | b_1 - a_1 | + \mu | b_2 - a_2 | < (\lambda + \mu)\delta = \delta$$

Since $a \in M$ (in virtue of the convexity of M), the last inequality implies that $b \in S(M, \delta)$, q.e.d.

Corollary. An open solid sphere $S(p, \delta)$ is a convex set.

The intersection of all the sets $S(M, \delta)$, $\delta > 0$ arbitrary, is the closure of M. Therefore, 2.22 and 2.23 imply

2.24. *The closure of a convex set is convex.*

Corollary. A closed solid sphere is a convex body.

§2.3. Interior and boundary points of a convex set.

2.31. *Let M be a convex set, a an arbitrary point of M, b an interior point of M (relative to $R^n \supseteq M$), c a point of the segment $[ab]$ different from a. Then c is an interior point of M.*

Proof. There exists a $\delta > 0$ such that $S(b, \delta) \subseteq M$; furthermore, $c = \lambda a + \mu b$, where $\lambda + \mu = 1$, $\lambda \geq 0$, and, since $c \neq a$, $\mu > 0$.

We shall prove that $S(c, \mu\delta) \subseteq M$.

If $d \in S(c, \mu\delta)$, i.e., if $| d - c | = | d - \lambda a - \mu b | < \mu\delta$, then

$$| (1/\mu)\, d - (\lambda/\mu)a - b | < \delta;$$

consequently the point

$$d' = (1/\mu)\, d - (\lambda/\mu)a$$

is contained in $S(b, \delta) \subseteq M$. Since

$$d = \lambda a + \mu d',$$

d is on the segment $[ad'] \subseteq M$, q.e.d.

COROLLARY. *The set of interior points of a convex set is convex.*

If a is a boundary point and b an interior point of M, then, in particular, by 2.31, $[ab]$ cannot contain any boundary point different from a. Hence

2.32. *Every half-line issuing from an interior point of a bounded convex set M contains one and only one boundary point of M.*

2.32 in turn implies

2.320. *If a convex set $M \subset R^n$ has an interior point, the set of all interior points of M is dense in M (and open in R^n).*

§2.4. **The dimension number** of a convex set Q is, by definition, the maximum number r such that Q contains $r + 1$ linearly independent points $a_0, \cdots, a_r$. If $a_0, \cdots, a_r$ are linearly independent in Q, Q is also contained in the plane $R^r(a_0, \cdots, a_r)$ (since if Q were to contain a point a not on this plane, Q would have $r + 2$ linearly independent points a, $a_0, \cdots, a_r$). The set of all points $x \in R^r(a_0, \cdots, a_r)$ whose barycentric coordinates with respect to $a_0, \cdots, a_r$ are positive is an open subset of $R^r(a_0, \cdots, a_r)$ contained in Q. Hence

2.41. *Every convex set of dimension number r is contained in a uniquely defined r-plane, the carrying plane of Q, and contains interior points with respect to this plane.*

Unless otherwise specified, *interior points of a convex set Q* will always refer to interior points of Q *relative to its carrying plane.* It follows from 2.31 that the interior points of a convex set Q form an r-dimensional convex set dense in Q.

2.42. *All n-dimensional convex bodies are homeomorphic.*

Proof. We shall prove that an n-dimensional convex body $\bar{Q}^n \subset R^n$ is homeomorphic to a closed solid n-sphere $\bar{E}^n$. Let us take as the center of $\bar{E}^n$ any interior point o of the convex set Q^n. Let x be an arbitrary point of $\bar{Q}^n$ different from o. Let us denote by $p(x)$ the boundary point of $\bar{Q}_n$ lying on the ray ox, by $q(x)$ the point in which the ray ox intersects the boundary of $\bar{E}^n$, and by $C(x)$ the point which divides the segment $[oq(x)]$ in the same ratio as the point x divides the segment $[op(x)]$. The result is a mapping C of the convex body $\bar{Q}^n$ onto $\bar{E}^n$; it is left to the reader to prove that this mapping is $(1-1)$ and bicontinuous (see, e.g., Alexandroff-Hopf [A-H; 601–602]).

§3. Closed convex hull. Simplexes. Convex polyhedral domains

§3.1. **The closed convex hull of a set** $M \subset R^n$ is defined as the intersection of all the convex sets containing M. In consequence of 2.22, it is a

convex set. It is obvious that M coincides with its closed convex hull if, and only if, M is convex.

3.11. *The diameter* (the least upper bound of the distances between two arbitrary points of a set) *of the closed convex hull M^* of a set M is equal to the diameter of M.*

Proof. It suffices to show that if δ is such that $|x - y| < \delta$ for any pair of points x, y of M, then $|a - b| < \delta$ for any pair of points a, b of M^*.

Let a, b be any two fixed points of M^* and let $x \in M$. Since $|x - y| < \delta$ for all $y \in M$, $M \subseteq S(x, \delta)$. In consequence of the convexity of $S(x, \delta)$ and the definition of closed convex hull, $M^* \subseteq S(x, \delta)$. Hence, in particular, $a \in S(x, \delta)$. Therefore $x \in S(a, \delta)$. This is true for any $x \in M$, so that $M \subseteq S(a, \delta)$. Hence $M^* \subseteq S(a, \delta)$ and, in particular, $b \in S(a, \delta)$. Therefore, $|a - b| < \delta$, q.e.d.

§3.2. Closed convex hulls of finite sets. Definition of a simplex and of a closed simplex.

3.21. The closed convex hull of a finite set of points $a_0, \cdots, a_s$ of R^n consists of all the points $a \in R^n$ of the form

$$a = \mu_0 a_0 + \cdots + \mu_s a_s, \tag{3.21}$$

where $\mu_0, \cdots, \mu_s$ are arbitrary nonnegative real numbers whose sum is 1.

In other words, the closed convex hull of a finite set $a_0, \cdots, a_s$ consists of the centroids of all possible nonnegative weights located at the points $a_0, \cdots, a_s$ respectively. In particular, if the points $a_0, \cdots, a_s$ are linearly independent (which implies that $s \leq n$), the closed convex hull of the set $a_0, \cdots, a_s$ consists of all the points of the plane $R^s(a_0, \cdots, a_s)$ whose barycentric coordinates with respect to $a_0, \cdots, a_s$ are nonnegative.

Proof of 3.21. A detailed proof can be found in Alexandroff-Hopf [A-H;602–604]. We shall give a sketch of this proof here. Denote the set of all points (3.21) by $\langle a_0, \cdots, a_s\rangle$ and the closed convex hull of the set $\{a_0, \cdots, a_s\}$ by $\{a_0, \cdots, a_s\}^*$.

It is required to prove that $\{a_0, \cdots, a_s\}^* = \langle a_0, \cdots, a_s\rangle$. To this end, we state the following three lemmas:

3.211. The set $\langle a_0, \cdots, a_s\rangle$ is convex.

3.212. If $0 \leq r \leq s$, and a is an arbitrary point of $\langle a_0, \cdots, a_s\rangle$, there exist points $a' \in \langle a_0, \cdots, a_r\rangle$ and $a'' \in \langle a_{r+1}, \cdots, a_s\rangle$ such that a is on the segment $[a'a'']$.

3.213. If the points $a_0, \cdots, a_s$ are contained in a convex set Q, then $\langle a_0, \cdots, a_s\rangle \subseteq Q$.

Lemma 3.211 can be verified by a routine calculation.

Lemma 3.212 is proved as follows. If $a = \mu_0 a_0 + \cdots + \mu_s a_s$, and if

$$\lambda' = \textstyle\sum_0^r \mu_i \neq 0, \qquad \lambda'' = \sum_{r+1}^s \mu_i \neq 0,$$

put

$$a' = \sum_0^r (\mu_i/\lambda')a_i$$

$$a'' = \sum_{r+1}^s (\mu_i/\lambda'')a_i .$$

Then $a = \lambda'a' + \lambda''a''$. But if, e.g., $\lambda' = 0$, then $a \in \langle a_{r+1}, \cdots, a_s\rangle$ and the lemma is trivial. Lemma 3.213 is proved by induction on the number s. If $a \in \langle a_0, \cdots, a_s\rangle$ and $a_0 \in Q, \cdots, a_s \in Q$, by Lemma 3.212 there is a point $a'' \in \langle a_1, \cdots, a_s\rangle$ such that $a \in [a_0a'']$. By the inductive hypothesis $\langle a_1, \cdots, a_s\rangle \subseteq Q$; since $a_0 \in Q$ and Q is convex, $a \in [a_0a''] \subseteq Q$, which was to be proved. Lemmas 3.211 and 3.213 immediately imply 3.21.

3.22. Let $a_0, \cdots, a_r$ be linearly independent points of R^n (so that $r \leq n$). The set of points of the plane $R^r(a_0, \cdots, a_r)$ whose barycentric coordinates with respect to $a_0, \cdots, a_r$ are positive is a convex open set in $R^r(a_0, \cdots, a_r)$ called an r-dimensional simplex or simply an *r-simplex* with vertices $a_0, \cdots, a_r$ and denoted by $(a_0 \cdots a_r)$. The closure of the simplex $(a_0 \cdots a_r)$, which is obviously synonymous with the closed convex hull of the set of points $a_0, \cdots, a_r$, is called the *closed simplex* with vertices $a_0, \cdots, a_r$ and is denoted by $[a_0 \cdots a_r]$.

If $a_{i_0}, \cdots, a_{i_p}$, $0 \leq p \leq r$, are vertices of a simplex $(a_0 \cdots a_r)$, the p-simplex $(a_{i_0} \cdots a_{i_p})$ is called a *p-face* of $(a_0 \cdots a_r)$. In particular, the r-simplex $(a_0 \cdots a_r)$ is its own unique r-face. The remaining faces will be called *proper* faces of the simplex.

Two faces $(a_{i_0} \cdots a_{i_p})$ and $(a_{j_0} \cdots a_{j_q})$ of a simplex $(a_0 \cdots a_r)$ are said to be *opposite* faces if every vertex of $(a_0 \cdots a_r)$ is a vertex of exactly one of the two faces. Obviously, if a p-simplex and a q-simplex are opposite faces of an r-simplex, then $p + q = r - 1$.

It is easily proved that

3.23. If $T^p = (a_{i_0} \cdots a_{i_p})$ and $T^q = (a_{j_0} \cdots a_{j_q})$ are opposite faces of a simplex $T^r = (a_0 \cdots a_r)$, then every point of T^r lies on exactly one straight line segment joining some point of T^p with some point of T^q.

Further, 3.11 implies

3.24. *The diameter of $(a_0 \cdots a_r)$ is equal to the maximum of the numbers $\rho(a_i, a_j)$.*

§3.3. Convex polyhedral domains. Since the half-spaces of a given R^n are convex sets, the intersection of any number of half-spaces (open or closed) is convex.

3.31. A bounded nonempty subset of R^n which is the intersection of a finite number of open (closed) half-spaces of R^n is called a *convex polyhedral domain* (*closed polyhedral domain*).

The closure of a convex polyhedral domain is a closed polyhedral do-

main of the same dimension number; conversely, the set of interior points of a closed convex polyhedral domain is a convex polyhedral domain.

Let Q^n be an n-dimensional convex polyhedral domain in R^n. The intersection of every $(n-1)$-plane $R^{n-1} \subset R^n$ with $\bar{Q}^n$ and Q^n is convex. A plane R^{n-1} is called a *plane of support* of the polyhedral domain Q^n if $\bar{Q}^n \cap R^{n-1} \neq 0$ and $R^{n-1} \cap Q^n = 0$.

The intersection of every supporting plane R^{n-1} with the boundary $\bar{Q}^n \setminus Q^n$ of the polyhedral domain Q^n coincides with the set $R^{n-1} \cap \bar{Q}^n$ and is therefore a closed convex polyhedral domain $\bar{Q}^r$, $r \leq n-1$; if $r = n-1$, Q^r is called an $(n-1)$-*face* of the polyhedral domain Q^n.

Hence *every $(n-1)$-dimensional polyhedral domain which is the interior of some closed polyhedral domain obtained as the intersection of $\bar{Q}^n$ with a supporting plane of Q^n is called an $(n-1)$-face of the convex polyhedral domain Q^n.*

Furthermore, the $(n-2)$-faces of the $(n-1)$-faces of Q^n are called the $(n-2)$-faces of Q^n, etc. The 0-faces of Q^n are points and are called vertices of Q^n.

3.32. *Every two faces of a convex polyhedral domain are disjoint; the union of all the r-faces of Q^n, $0 \leq r \leq n-1$, is the boundary $\bar{Q}^n \setminus Q^n$ of Q^n.*

REMARK. A polyhedral domain Q^n is its own unique n-face.

The proof of Theorem 3.32, which offers no serious difficulties but is nevertheless quite long and tiresome (see Alexandroff-Hopf [A-H; 609–614]), is left to the reader. We note finally that the least possible number of vertices of an n-dimensional convex polyhedral domain is $n+1$. Convex polyhedral domains which have exactly $n+1$ vertices are n-simplexes.

§4. Centroid of a simplex

DEFINITION 4.1. *The centroid* of a finite set of points $a_1, \cdots, a_k$ of R^n is the point

$$b = (1/k)(a_1 + \cdots + a_k),$$

i.e., the centroid of the system of equal weights located at these points.

The centroid of a simplex (and in general of a convex polyhedral domain) is, by definition, the *centroid of its set of vertices* in the above sense.

§4.2. Let $M \subset R^n$ be a finite set of diameter d consisting of k points, let M_1 be a nonempty subset of M, and denote by b and b_1 the centroids of M and M_1, respectively. Then

$$\rho(b, b_1) \leq [(k-1)/k]\, d. \tag{4.20}$$

Indeed, if

$$M = \{a_1, \cdots, a_k\}, \qquad M_1 = \{a_1, \cdots, a_{k_1}\}, \qquad (1 \leq k_1 \leq k),$$

then, by the definition of centroid,

$$b = (1/k) \sum_{i=1}^{k} a_i , \qquad b_1 = (1/k_1) \sum_{j=1}^{k_1} a_j ,$$

so that

$$b_1 - b = (1/k_1)(\sum_{j=1}^{k_1} a_j - k_1 b) = (1/k_1) \sum_{j=1}^{k_1} (a_j - b),$$

$$a_j - b = (1/k)(ka_j - \sum_{i=1}^{k} a_i) = (1/k) \sum_{i=1}^{k} (a_j - a_i),$$

and

$$b_1 - b = (1/kk_1) \sum_{i=1}^{k} \sum_{j=1}^{k_1} (a_j - a_i). \tag{4.21}$$

The kk_1 summands $a_j - a_i$ contain k_1 summands such that $j = i \leq k_1$ and these summands are equal to zero. This means that the number of summands different from zero in the sum $\sum_{i=1}^{k} \sum_{j=1}^{k_1} (a_j - a_i)$ is equal to $kk_1 - k_1 = k_1(k - 1)$. Therefore, (4.21) and the assumption that $| a_j - a_i | \leq d$ imply that

$$| b_1 - b | \leq (1/kk_1)\, k_1(k - 1)\, d = [(k - 1)/k]\, d, \qquad \text{q.e.d.}$$

Corollary 4.2. Let T^n be an n-simplex of diameter $<d$, let T' be a face of arbitrary dimension of T^n, and let T'' be a face of T'. Denoting the centroids of T', T'' by b', b'', respectively, we have

$$\rho(b', b'') \leq [n/(n + 1)]\, d.$$

Indeed, if the dimension of T' is r, in consequence of the above, $\rho(b', b'') \leq [r/(r + 1)]\, d$. But $r/(r + 1) \leq n/(n + 1)$, whence the assertion also follows.

§5. Central projection

Let o be a point of R^n, called a *center of projection*. Let $M \subseteq R^n \setminus o$. We shall call the point set $oM([oM])$ of R^n which is the union of all open segments (ox) (closed segments $[ox]$), $x \in M$, the *open* (*closed*) *projection* of the set M, or cone with base M and vertex o.

Let $N \subset R^n \setminus o$ be any set intersecting every half-line joining o with any point $x \in M$ in a single unique point $\pi(x)$. The assignment to each $x \in M$ of the point $\pi(x) \in N$ yields a mapping π of the set M into the set N.

The mapping π, as well as the image $\pi(M)$ of M under this mapping, is called the *projection* of M *into* N *from* o. When there can be no misunderstanding, we shall use the term projection to denote a projection as a mapping.

The proof of the following property of projections is left to the reader:

5.1. *Suppose that the intersection of a set M with every half-line issuing from o consists of a single point and let M_1, M_2 be two subsets of M. Then the projection of the intersection of M_1 and M_2 coincides with the intersection of their projections. By projection is meant either open or closed projection.*

Kazan, December 10, 1941.

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